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## Edge-colourings of graphs

### Thesis

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EDGE-COLOURINGS OF GRAPHS

by

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A thesis submitted in fulfilment of  
the requirements for the degree of  
Doctor of Philosophy

Department of Mathematics

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### Abstract

All the results in this thesis are concerned with the classification of graphs by their chromatic class.

We first extend earlier results of Fiorini and others to give a complete list of critical graphs of order at most ten. We give conditions for extending the edge-colouring of a nearly complete subgraph to the whole graph and use this result to prove a special case of Vizing's conjecture. We also use other methods to solve further cases of this conjecture.

A major part of the thesis classifies those graphs with at most 4 vertices of maximum degree and this work is generalised to graphs with  $r$  vertices of maximum degree. We also completely classify all regular graphs  $G$  with degree at least  $\frac{6}{7}|V(G)|$ .

Finally we give some examples of even order critical graphs and introduce the concept of a supersnark.



## Preface

This thesis is the outcome of my research work as a Ph.D student at the Open University.

I should like to record my warmest thanks to my supervisor, Dr. A.J.W. Hilton, firstly for his ready acceptance of my studentship at such a late stage and then for the substantial encouragement and the intensive direction he gave to the completion of my research. I should also like to thank Dr. R.J. Wilson for his help at the outset of my studies.

Most of this thesis is the result of this close collaboration with Dr. Hilton. However the results in Chapter 3 of this thesis are the product of joint work done with Dr. H.P. Yap. Lemma 8.2 was developed with the help of Dr. L.D. Andersen.

Most of the material in this thesis has been submitted for publication. A survey of known results on snarks appears in:

A.G. Chetwynd and Robin J. Wilson    Snarks and Supersnarks. In the  
Theory and Applications of Graphs,  
Wiley, New York (1981).

The counter-examples to the critical graph conjecture appear in:

A.G. Chetwynd and Robin J. Wilson    The Rise and Fall of the Critical  
Graph Conjecture, Journal of Graph  
Theory,    7 (1983), 153-157.

The results of Chapter 3 appear in:

A.G. Chetwynd and H.P. Yap    Chromatic Index Critical Graphs of  
Order 9, Discrete Mathematics,  
47    1 (1983), 23-33.

Chapters 4,5 and 6 are to appear in the following two papers:

- |                                 |   |
|---------------------------------|---|
| A.G. Chetwynd and A.J.W. Hilton | Partial edge-colourings of complete graphs or of graphs which are nearly complete, Proc. of Comb. Conference at Cambridge, 1983, to appear. |
| A.G. Chetwynd and A.J.W. Hilton | The Chromatic index of graphs of even order with many edges, Journal of Graph Theory, to appear.  |

The results of Chapter 9 are to appear in:

- |                                 |   |
|---------------------------------|---|
| A.G. Chetwynd and A.J.W. Hilton | Regular graphs of high degree are 1-factorizable, Proceedings of the London Mathematics Society, to appear. |
|---------------------------------|---|

Further papers are in preparation containing the contents of Chapters 7,8 and some of Chapter 10.

During my studentship I attended conferences in Waterloo, Vancouver, Swansea, Cambridge and Southampton, where I had many useful discussions about the development of this thesis. In particular I should like to thank: L.D. Andersen, S. Fiorini, R. Häggkvist, B. Jackson, W.T. Trotter and J. Watkins.

Finally I should like to thank the Open University for their financial support throughout my studentship.

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## 1. Introduction

The idea of edge-colouring a graph was first introduced by P.G. Tait [T1] in 1880 in an attempt to prove the Four Colour Conjecture. The first major result on edge-colourings was in 1964 when V.G. Vizing proved that every simple graph  $G$  with maximum degree  $\Delta$  can be properly edge-coloured with at most  $\Delta+1$  colours. Vizing's result partitions the set of all graphs into two disjoint classes. The first, Class 1, consists of those graphs of maximum degree  $\Delta$  which can be edge-coloured with  $\Delta$  colours, whereas the second, Class 2, consists of those graphs that require  $\Delta+1$  colours.

The general problem of classifying all graphs is extremely difficult as can be seen since a solution would have as a corollary the four colour theorem.

This thesis is mainly concerned with the classification problem for different types of graphs and looks at some of the particular graphs which require  $\Delta+1$  colours. We tackle this problem from several different angles.

We first look at small order graphs and extend results of Fiorini and others to classify all graphs on at most 10 vertices. Our second approach classifies graphs whose degree is high relative to the order. The first such graphs we consider are obtained from a complete graph by removing a few edges. We obtain results on these graphs by giving necessary and sufficient conditions for an edge-colouring of an induced subgraph of  $G$  to be extended to an edge-colouring of  $G$  with  $\chi'(G)$  colours.

By adapting the proof of Vizing's theorem to multigraphs we have been able to show for odd order graphs that if a graph  $G$  has  $2n+1$  vertices,  $\binom{2n+1}{2} - 2n$  edges, and maximum degree  $2n-1$ , then it is Class 1. For even order graphs we have found a necessary and sufficient condition for a

graph with  $2n+2$  vertices and maximum degree  $2n-1$  to be Class 1. Both these results answer problems posed by Plantholt.

Our third approach considers  $r$ , the number of vertices of maximum degree. We have been able to classify all graphs with  $1 \leq r \leq 4$  and all graphs with  $\Delta \geq n + \frac{7}{2}r - 3$ .

We have also shown that regular graphs of even order with  $d(G) \geq \frac{6}{7} |V(G)|$  are Class 1. This result is a partial solution to the conjecture that all regular graphs of even order which satisfy  $d(G) \geq \frac{1}{2} |V(G)|$  are Class 1.

The results mentioned in the previous two paragraphs are the most significant results of the Thesis.

We introduce the concept of super-snarks and give examples. Goldberg gave a 3-critical counter-example to the conjecture that all critical graphs had odd order. We give some 4-critical counter-examples. We also exhibit a family of graphs which are obtained from the double star snark and show which of these are Class 2.

## 2. Definitions and known results.

We now give the basic definitions and results on edge-colourings used in this thesis.

We denote the maximum degree of a graph  $G$  by  $\Delta(G)$  and the minimum degree by  $\delta(G)$ . The degree of a particular vertex  $v$  is  $d(v)$ . Vertices adjacent to a vertex  $v$  are called its neighbours and the set of neighbours of  $v$  is denoted by  $N(v)$ . We let  $d^*(v)$  be the number of neighbours of  $v$  of maximum degree.

The order of  $G$  is the number of vertices of  $G$  and is denoted by  $|V(G)|$ . The size of  $G$  is the number of edges of  $G$  and is denoted by  $|E(G)|$ .

If  $W$  is a set of vertices of  $G$ , then  $G \setminus W$  is the graph  $G'$  such that  $V(G') = V(G) \setminus W$  and  $E(G') = E(G) \setminus \{wx : w \in V(G) \text{ and } x \in W\}$ . Similarly if  $M$  is a set of edges of  $G$ , then  $G \setminus M$  is the graph  $G''$  such that  $V(G'') = V(G)$  and  $E(G'') = E(G) \setminus M$ . If  $Y \subset V(G)$  then  $\langle Y \rangle$  denotes the subgraph of  $G$  induced by  $Y$ .

The deficiency of  $G$  is the sum

$$\sum_{v \in V(G)} (\Delta(G) - d(v)).$$

and is denoted by  $\text{def}(G)$ .

Two graphs  $G$  and  $H$  are said to be isomorphic if there exists a one to one correspondence between  $V(G)$  and  $V(H)$  which preserves adjacency. We then write  $G \cong H$ .

We define an edge-colouring of  $G$  to be a mapping  $\phi: E(G) \rightarrow \mathcal{C}$  where  $\mathcal{C}$  is a set of colours such that if  $e_1$  and  $e_2$  are edges with a common vertex then  $\phi(e_1) \neq \phi(e_2)$ . The least number  $j$  for which there exists an

edge-colouring of  $G$  with  $j$  colours is called the chromatic index of  $G$  and is denoted by  $\chi'(G)$ .

Vizing [V1] showed that if  $G$  is a simple graph then

$$\Delta(G) \leq \chi'(G) \leq \Delta(G)+1.$$

$G$  is Class 1 if  $\chi'(G) = \Delta(G)$  and is Class 2 if  $\chi'(G) = \Delta(G)+1$ . If  $G$  is a multigraph then  $G$  is Class 1 if  $\chi'(G) = \Delta(G)$ , and is Class 2 otherwise.

If  $G$  is Class 2 and  $\chi'(G') < \chi'(G)$  for all proper subgraphs  $G'$  of  $G$ , then  $G$  is said to be critical.

A set of independent edges is called a matching. A matching covering all vertices is called a 1-factor. A near 1-factor is a matching covering all but one of the vertices. A regular, Class 1 graph is often called 1-factorizable as it is the union of edge disjoint 1-factors.

If a graph is edge-coloured and a colour  $c$  is present on one of the edges incident with some vertex  $v$ , we say that  $c$  is present at  $v$ . If  $c$  is not present on any of the edges incident with  $v$ , then we say that  $c$  is absent at  $v$ .

If  $x$  is a real number, then  $\lfloor x \rfloor$  denotes the largest integer not greater than  $x$  and  $\lceil x \rceil$  denotes the smallest integer not less than  $x$ .

If  $x_1 \leq x_2 \leq \dots \leq x_\ell$  and a graph  $G$  has  $r_i$  vertices of degree  $x_i$  for  $1 \leq i \leq \ell$ , then we write

$$G \equiv \overset{r_1}{x_1} \overset{r_2}{x_2} \dots \overset{r_\ell}{x_\ell}.$$

Other basic graph theoretic terminology can be found in any standard introduction to the subject such as [B3].

Our first Lemma is Vizing's Adjacency Lemma [V2] and will be used frequently.

Lemma 2.1    Let  $G$  be a critical graph. Let  $u, w \in V(G)$  and let  $u$  be adjacent to  $w$ . Then



$$d^*(w) \geq \begin{cases} \Delta(G) - d(u) + 1 & \text{if } d(u) < \Delta(G), \\ 2 & \text{if } d(u) = \Delta(G). \end{cases}$$

An accessible proof of this lemma can be found in [F6]. An immediate corollary of Lemma 2.1 is

Lemma 2.2. Let  $G$  be a critical graph. Then each vertex is adjacent to at least two vertices of maximum degree (i.e.  $d^*(v) \geq 2 \quad \forall v \in V(G)$ ).

Lemma 2.3 [V2] Let  $G$  be a graph of Class 2 with maximum degree  $\Delta$ . Then  $G$  contains a critical subgraph of maximum degree  $k$  for each  $k$  satisfying  $2 \leq k \leq \Delta$ .

For our purposes, the following result is extremely useful.

Lemma 2.4 For a graph  $G$ , let  $e \in E(G)$  and  $w \in V(G)$ , and let  $e$  and  $w$  be incident. Let  $d^*(w) \leq 1$ . Then

$$\begin{aligned} \Delta(G \setminus e) &= \Delta(G) \Rightarrow \chi'(G \setminus e) = \chi'(G) \text{ and} \\ \Delta(G \setminus w) &= \Delta(G) \Rightarrow \chi'(G \setminus w) = \chi'(G). \end{aligned}$$

Proof. If  $G$  is Class 1 then we have

$$\begin{aligned} \Delta(G) &= \chi'(G) \geq \chi'(G \setminus e) \geq \Delta(G \setminus e) = \Delta(G), \\ \text{so } \chi'(G) &= \chi'(G \setminus e), \text{ and similarly} \\ \Delta(G) &= \chi'(G) \geq \chi'(G \setminus w) \geq \Delta(G \setminus w) = \Delta(G), \\ \text{so } \chi'(G) &= \chi'(G \setminus w). \end{aligned}$$

If  $G$  is Class 2, let  $G^*$  be a critical subgraph of  $G$  with  $\Delta(G) = \Delta(G^*)$ . Then, in view of Lemma 2.1,  $e \notin E(G^*)$  because  $d_{G^*}^*(w) \leq d_G^*(w) \leq 1$ . Similarly  $w \notin V(G^*)$ .

There is an alternative proof of Lemma 2.4 which does not depend on Vizing's Adjacency Lemma, nor on the notion of critical graphs. It does however depend on knowledge of the original proof of Vizing's theorem that  $\chi'(G) \leq \Delta(G) + 1$ .

Alternative proof of Lemma 2.4. If  $d^*(w) \leq 1$  and  $\Delta(G \setminus w) = \Delta(G)$  then Vizing's argument may be applied to extend the edge-colouring of  $G \setminus w$ , without increasing the chromatic index, provided that  $w$  is the pivot vertex and the edge (if there is one) joining  $w$  to a vertex of degree  $\Delta(G)$  is coloured last. If  $d^*(w) \leq 1$ ,  $\Delta(G \setminus e) = \Delta(G)$  and  $\Delta(G \setminus w) = \Delta(G)$  then  $\chi'(G \setminus e) = \chi'(G)$ , since  $\chi'(G) \geq \chi'(G \setminus e) \geq \chi'(G \setminus w) = \chi'(G)$ .

If however  $d^*(w) \leq 1$ ,  $\Delta(G \setminus e) = \Delta(G)$  and  $\Delta(G \setminus w) \neq \Delta(G)$  then  $G$  and  $G \setminus e$  have just one (the same) vertex of maximum degree other than possibly  $w$ . It is easy to show by Vizing's argument that both are Class 1, provided an edge on this vertex of maximum degree is coloured last. (If  $d(w) = \Delta$  colour the edge joining the two vertices of maximum degree last.)

Lemma 2.5    If  $G$  has 1 or 2 vertices of maximum degree, then  $G$  is Class 1.

Proof.        If  $G$  is not Class 1 then  $G$  has a critical subgraph  $G^*$  with  $\Delta(G) = \Delta(G^*)$ , and  $G^*$  has at most two vertices, say  $u, w$  of maximum degree. But  $d^*(u) \leq 1$  which contradicts Lemma 2.2. Therefore  $G$  is Class 1.

Lemma 2.6    Let  $G$  be a critical graph. If  $G$  has  $r$  vertices of degree  $\Delta(G)$ , then

$$\delta(G) \geq \Delta(G) - r + 2.$$

Proof.        Let  $u$  be a vertex with  $d(u) = \delta(G)$  and let  $w$  be a vertex adjacent to  $u$  of degree  $\Delta(G)$  [there is such a vertex by Lemma 2.2]. By Lemma 2.1,  $w$  is adjacent to at least  $\Delta(G) - d(u) + 1$  vertices of maximum degree. Thus including  $w$ , there are at least  $\Delta(G) - \delta(G) + 2$  vertices of degree  $\Delta(G)$ , so  $r \geq \Delta(G) - \delta(G) + 2$ , whence the result.

Lemma 2.7    Let  $G$  be a graph with  $v$  vertices and  $e$  edges, and with maximum degree  $\Delta$ ; then  $G$  is of Class 2 if  $e > \Delta \left\lfloor \frac{v}{2} \right\rfloor$ .

Proof.        If  $G$  is Class 1, then each colour class can have at most

$\left\lfloor \frac{v}{2} \right\rfloor$  edges. Since there are  $\Delta$  colours the maximum number of edges is  $\Delta \left\lfloor \frac{v}{2} \right\rfloor$ .

The following results may be found in [F6].

Lemma 2.8 Let  $G$  be a critical graph. If  $G$  has maximum degree  $\Delta$  and if  $v$  and  $w$  are adjacent vertices of  $G$ , then  $d(v) + d(w) \geq \Delta + 2$ .

Lemma 2.9 Let  $G$  be a critical graph with  $v$  vertices and  $e$  edges, maximum degree  $\Delta$  and minimum degree  $\delta$  then

$$\begin{aligned} e &\leq \frac{1}{2}(v-1)\Delta + 1 && \text{if } v \text{ is odd,} \\ e &\leq \frac{1}{2}(v-2)\Delta + \delta - 1 && \text{if } v \text{ is even.} \end{aligned}$$

Lemma 2.10 Let  $G$  be a critical graph of maximum degree  $\Delta$  and let  $n_j$  be the number of vertices of degree  $j$  for  $j = 2, 3, \dots, \Delta$ . Then for each  $k$  satisfying  $2 \leq k \leq \Delta-1$ , we have

$$\sum_{j=2}^k \frac{n_j}{j-1} < \frac{n_{\Delta}}{2}.$$

Lemma 2.11 A critical graph contains no cut vertices.

The next lemma is due to König [K1].

Lemma 2.12 If  $G$  is a bipartite graph with maximum vertex degree  $\Delta$ , then  $\chi'(G) = \Delta$ .

The next lemma is Tutte's theorem [T2].

Lemma 2.13. A graph  $G$  has a 1-factor if and only if, for all  $S \subset V(G)$ , the number of odd components in  $G-S$  is not more than  $|S|$ .

### 3. Critical graphs of order 9.

#### 3.1 Introduction and definitions.

In this chapter we extend the results of Beineke, Fiorini and Jakobsen to give a complete catalogue of all the chromatic index critical graphs of order  $\leq 10$ . In particular Jakobsen [J3] has constructed all 3-critical graphs of order  $\leq 10$ , Beineke and Fiorini [B1] have constructed all critical graphs of order 7 and Fiorini [F4] has shown that there are no critical graphs of even order  $\leq 10$ .

We begin by finding the 4-critical graphs of order 9 and end up with 8-critical graphs of order 9. The degree of difficulty for finding the degree-lists of  $\Delta$ -critical graphs decreases as  $\Delta$  increases because each time we use the results obtained previously and because the number of cases to be examined decreases as  $\Delta$  increases. In the subsequent sections, if  $\pi$  is a  $k$ -colouring of  $G$  and  $C_1, \dots, C_k$  are the colour classes of  $E(G)$  with respect to  $\pi$ , we always assume that for each  $f \in C_i$ ,  $\pi(f) = i$  and  $|C_1| \geq |C_2| \geq \dots \geq |C_k|$ .

We shall need the following lemmas.

Lemma 3.1    For  $\Delta \geq 3$ , there does not exist a  $\Delta$ -critical graph with a vertex  $u$  of degree 2, a vertex  $v (\neq u)$  of degree  $< \Delta$  and all other vertices of degree  $\Delta$ .

Proof.        Suppose such a  $\Delta$ -critical graph  $G$  exists and let  $u$  be joined to  $u_1$  and  $u_2$ . Since  $G$  is critical,  $G \setminus \{uu_1\}$  is Class 1 and can therefore be coloured with  $\Delta$ -colours.

Vertex  $u$  has 1 colour,  $u_1$  has  $\Delta-1$  colours,  $v$  has  $d(v)$  colours, and all other vertices have  $\Delta$  colours. The colour at  $u$  must be the colour missing at  $u_1$ , or else we could colour  $uu_1$ . Therefore there are  $\Delta$  colours missing at  $u$  and  $u_1$ , and  $\Delta-d(v)$  colours missing at  $v$ . Since

$d(v) \neq \Delta$  some colours will be missing an odd number of times and some an even number of times. This is impossible and hence  $G \setminus \{uu_1\}$  is not Class 1. Therefore  $G$  is not critical.

Lemma 3.2    If there exists a  $\Delta$ -critical graph  $G$  such that  $n_4 = 2n_2$  and  $n_3 = 0$ , then, for each vertex  $x$  of degree 2 in  $G$ , the two vertices adjacent to  $x$  must be adjacent.

Lemma 3.3    If  $G$  is a  $\Delta$ -critical graph of order  $n$  on  $m$  edges then

a)  $m \geq 2n+1$                       if  $\Delta = 5$ ,

b)  $m \geq \left(\frac{9n+1}{4}\right)$                       if  $\Delta = 6$ ,

c)  $m \geq \frac{5n}{2}$                       if  $\Delta = 7$ .

Proofs of Lemmas 3.2 and 3.3 can be found in Yap [Y2].

We now give a catalogue of the degree lists of all the critical graphs on at most 7 vertices and the 3-critical graphs on 9 vertices. This was found by Beineke and Fiorini [B1] and Jakobsen [J3]. It should be noted that all 2-connected graphs with the degree lists given, with the exception of the degree lists  $2^3 3^6$  and  $2^3 3^8$ , are critical.

### Catalogue

$v \backslash \Delta$	2	3	4	5	6
3	$2^3$				
5	$2^5$	$23^4$	$3^2 4^3$		
7	$2^7$	$23^6$	$24^6$ $3^2 4^5$	$25^6$ $345^5$ $4^3 5^4$	$45^2 6^4$ $5^4 6^3$
9	$2^9$	$23^8$ $2^3 3^6$ *			

\* The only graph which is critical with degree list  $2^3 3^6$  is the graph  $P^*$  drawn in Figure 3.1. Clearly any graph  $23^8$  which contains  $P^*$  as a subgraph is not critical

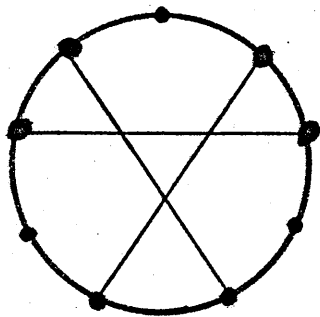


Figure 3.1. The Graph  $P^*$ .

### 3.2 4-Critical Graphs

We first prove

Lemma 3.4    If  $G$  is a  $\Delta$ -critical graph of odd order and  $F$  is a 1-factor of  $G$  where  $d(x) < \Delta$ , then  $G \setminus F$  has a  $(\Delta - 1)$ -critical subgraph  $H$ .

Proof.         $\chi'(G \setminus F) = \Delta$ , otherwise any  $(\Delta - 1)$ -colouring of  $G \setminus F$  can be extended to a  $\Delta$ -colouring of  $G$ . By the choice of  $x$ , the maximum degree of  $G \setminus F$  is  $\Delta - 1$ . Hence  $G \setminus F$  is Class 2. Lemma 3.4 now follows from Lemma 2.3.

We now prove

Theorem 3.1    A 2-connected graph  $G$  of order 9 is 4-critical if and only if its degree-list is either  $24^8$  or  $3^2 4^7$ , except for the graph of Figure 3.6.

Proof.        By Lemma 2.11, a critical graph must be 2-connected. Let  $G$  be a 4-critical graph of order 9 having minimum degree  $\delta$ .

Suppose  $\delta = 2$ . By Lemmas 2.1 and 2.10,  $n_4 \geq \max \{4, 2n_2 + n_3\}$ .

If  $n_3 = 0$ , then the possible degree lists of  $G$  are:  $2^3 4^6$ ,  $2^2 4^7$  and  $24^8$ .

If  $n_3 \neq 0$ , then the possible degree-list of  $G$  is  $23^2 4^6$ .

Suppose  $\delta = 3$ . By Lemma 2.10,  $n_3 \leq n_4$ . Hence  $3^2 4^7$  and  $3^4 4^5$  are

the only two possible degree-lists for  $G$ .

By Lemma 3.1 there are no critical graphs having degree list  $2^2 4^7$ .

Theorem 3.1 now follows from the following five lemmas:

Lemma 3.5    There are no critical graphs having degree-list  $2^3 4^6$ .

Proof.    Suppose such a critical graph  $G$  exists. Let  $x_1, x_2$  and  $x_3$  be vertices of degree 2. By Lemma 2.1,  $N(x_i) \cap N(x_j) = \emptyset$ , if  $i \neq j$ . Let  $N(x_i) = (y_1, y_2)$ ,  $N(x_2) = (z_1, z_2)$ ,  $N(x_3) = (w_1, w_2)$ . By Lemma 3.2,  $y_1 y_2, z_1 z_2, w_1 w_2 \in E(G)$ .

Let  $\pi$  be a 4-colouring of  $G \setminus x_1 y_2$ . Since  $|E(G \setminus x_1 y_2)| = 14$ , two of the four colour classes  $E_1, \dots, E_4$ , say  $E_1$  and  $E_2$ , of  $G \setminus x_1 y_2$  with respect to  $\pi$  must be of cardinality 4. Hence  $G \setminus x_1$  has a 1-factor  $F = E_1$  or  $E_2$ , so that  $\Delta(G \setminus F) = 3$  and  $\chi'(G \setminus F) = 4$  and, by Lemma 3.4,  $G \setminus F$  has a 3-critical subgraph  $H$ .

Suppose  $|V(H)| = 9$ . A 3-critical graph of order 9 does not have any vertices of degree 1. Therefore  $H$  is not a subgraph of  $G \setminus F$ , since  $G \setminus F$  has two vertices of degree 1.

Suppose  $|V(H)| = 7$ . Then  $|E(H)| = 10$  from the catalogue. Since  $H$  has only one vertex of degree 2 we may assume that  $x_2, x_3 \notin V(H)$ . Then  $|E(H)| \leq 15 - 4 - 4 = 7$  which yields a contradiction.

Suppose  $|V(H)| = 5$ . Then  $|E(H)| = 7$  from the catalogue. Since  $H$  has only one vertex of degree 2, we may assume that  $x_2, x_3 \notin V(H)$ . Then  $|E(H)| \leq 15 - 4 - 5 = 6$  which yields a contradiction.

Lemma 3.6    There are no critical graphs having degree-list  $23^2 4^6$ .

Proof.    Suppose such a critical graph  $G$  exists. Then  $|E(G)| = 16$ . Let  $x, y, z \in V(G)$  be such that  $d(x) = 2$ ,  $d(y) = d(z) = 3$  and let  $x_1, x_2$  be the neighbours of  $x$ . By Lemma 2.1,  $y, z \notin N(x_1) \cup N(x_2)$ .

Let  $\pi$  be a 4-colouring of  $G \setminus e$  where  $e \in E(G)$ , let  $E_1, \dots, E_4$  be the colour classes of  $E(G \setminus e)$  with respect to  $\pi$ , and let  $|E_1| = |E_2| = |E_3| = 4$  and  $|E_4| = 3$ .

We first prove that  $yz \notin E(G)$ . Suppose otherwise. Then  $x, y, z$  are the only vertices of  $G \setminus yz$  having degree  $< 4$ . Hence, in the above colouring  $\pi$  of  $G \setminus e$  where  $e = yz$ , colour 4 is not present at all these three vertices. But then  $\pi$  can be extended to a 4-colouring of  $G$ , contradicting the hypothesis.

Next, we consider the above  $\pi$  colouring of  $G \setminus e$  where  $e = zw$ . Without loss of generality, we may assume that colour 1 is missing at  $z$ . Then  $F = E_1$  forms a near 1-factor of  $G$  and  $G \setminus F$  has valency-list  $123^7$ . By Lemma 3.1,  $G \setminus F$  has a 3-critical subgraph  $H$ . It is obvious that  $|V(H)| \neq 9$  and since there are no critical graphs of even order  $\leq 10$ ,  $|V(H)| = 7$  or  $5$ .

Suppose  $|V(H)| = 7$ . Then  $H$  has degree-list  $23^6$  (see catalogue). Let  $Y = V(G) \setminus V(H)$ . Then  $x \in Y$ . Since  $G \setminus F$  has 12 edges and  $H$  has 10 edges,  $x$  must be joined to the vertex of degree 2 in  $G \setminus F$ . But then in  $G$ ,  $x$  will have degree 2 and be joined to a vertex of degree at most 3 and this contradicts Lemma 2.1. Hence  $|V(H)| \neq 7$ .

Suppose  $|V(H)| = 5$ . Then  $H$  has degree-list  $23^4$  (see catalogue). Again, let  $Y$  be defined as above. Then  $x \in Y$  and  $|E(\langle Y \rangle)| \leq 5$ . Let  $I$  be the set of edges of the graph induced by  $Y$  in  $G \setminus F$ . There are at most 4 edges in  $I$  since  $x$  has degree 1 in  $G \setminus F$ .  $G \setminus F$  has 12 edges and  $H$  has 7 edges and there is at most one edge from  $V(H)$  to  $V(I)$  in  $G \setminus F$ . Hence  $I$  has at least 4 edges, so  $|I| = 4$  in  $G \setminus F$ .

Let  $x$  be joined to  $t$  and let  $t$  have neighbours  $z_1, z_2$  in  $G \setminus F$ . One of the  $z_i$  will have degree at most 3 in  $G$  and therefore Lemma 2.1 is



contradicted for the vertex  $t$  in  $G$ .

The proof of Lemma 3.6 is complete.

Lemma 3.7    Any 2-connected graph  $G$  having valency-list  $24^8$  is critical.

Proof.    The size of  $G$  is  $17 > 4 \left\lfloor \frac{9}{2} \right\rfloor$ . By Lemma 2.7,  $G$  is Class 2. If  $G$  is not critical, then by Lemma 2.3,  $G$  contains a 4-critical subgraph  $H$ . Since there is no critical graph of even order  $\leq 10$  and we have shown above that there is no 4-critical graph of order 9 with minimum degree 2 and having smaller size, the order of  $H$  must be 7 or 5.

We now prove that  $G$  cannot contain a 4-critical subgraph of order 7 or 5.

Suppose the graph  $K$  is obtained from  $G$  by deleting two vertices. The number of vertices of degree 4 in  $K$  is at most 4. But a 4-critical graph of order 7 has degree-list either  $24^6$  or  $3^2 4^5$  (see catalogue). Hence  $G$  cannot contain a 4-critical subgraph of order 7.

Finally there is only one 4-critical graph of order 5, namely  $K_5 \setminus e$  (see catalogue). But  $K_5 \setminus e$  cannot be extended to a graph of order 9 having degree-list  $24^8$ .

Lemma 3.8    There are no critical graphs having degree-list  $3^4 4^5$ .

Proof.    Suppose such a critical graph  $G$  exists. Let  $X = \{x, y, u, v\}$  be the set of vertices of degree 3 and let  $K = \langle X \rangle$ . If  $K = 0_4$ , the graph of order 4 and size 0, then there is at least one vertex  $z$  of degree 4 adjacent to three or more vertices in  $X$ ; but this is prohibited by Lemma 2.1. Again, by Lemma 2.1 it is clear that  $K = K_2 \cup 0_2$  or  $2K_2$ . Hence we may assume that  $xy \in G$ .

Let  $z$  be of degree 4 and adjacent to  $u$ . Let  $\pi$  be a 4-colouring of  $G \setminus uz$  and let  $E_1, \dots, E_4$  be the four colour classes of  $G \setminus uz$  with respect to  $\pi$  such that

$|E_1| > \dots > |E_4|$ . Since  $|E(G \setminus uz)| = 15$ , we have  $|E_1| = |E_2| = |E_3| = 4$  and  $|E_4| = 3$ . It is clear that  $\chi'(G \setminus E_i) = 4$  for each  $i = 1, 2, 3, 4$ . We note that the degree-list of  $G \setminus uz$  is  $23^4 4^4$  and that, for each  $i = 1, 2, 3$ , there is exactly one vertex of degree  $< 4$  in  $G \setminus uz$  at which colour  $i$  is not present.

We now show that there is  $i = 1, 2$  or  $3$  such that  $G \setminus E_i$  is Class 2 and cannot be 3-critical. From this it follows that, for some  $i = 1, 2$  or  $3$ ,  $G \setminus E_i$  has a 3-critical subgraph  $H$  of order 7 or 5. We will finally prove that this is not possible and hence there are no critical graphs having degree-list  $3^4 4^5$ .

Applying the fact that there is only one 3-critical graph of order 9 and size 12 (see catalogue) namely  $P^*$ , we can prove by examining all cases that there is  $i = 1, 2$  or  $3$  such that  $G \setminus E_i$  is of Class 2 and cannot contain a 3-critical subgraph of order 9. Hence, for some  $i = 1, 2$  or  $3$ ,  $G \setminus E_i$  contains a 3-critical subgraph  $H$  of order 7 or 5. From now on, we write  $F$  for this particular  $E_i$ .

Suppose  $|V(H)| = 7$ . Let  $V(G) \setminus V(H) = \{v_1, v_2\}$ . In this case, the degree-list of  $H$  is  $23^6$  (see catalogue). Since  $F$  is a near 1-factor of  $G$ ,  $F$  has at least two but not more than three edges in  $\bar{H}$ . Let  $B$  be the set of edges joining  $V(H)$  with  $\{v_1, v_2\}$ . Then  $H$  has deficiency  $\leq 6$ , so  $|B| \leq 2$  and  $\min\{d(v_1), d(v_2)\} \leq 2$ , which is false.

Suppose  $|V(H)| = 5$ . Then  $H$  is the graph given in Figure 3.2. Let  $Y = V(G) \setminus V(H)$  and let  $I = \langle Y \rangle$ . Let  $J$  be the subgraph of  $G$  induced by  $V(H)$ . We note that since  $G$  is connected,

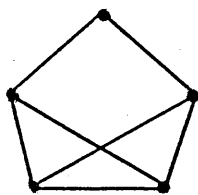


Figure 3.2

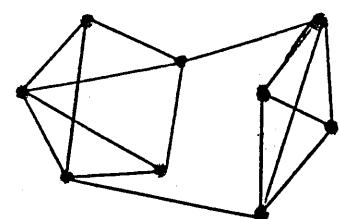
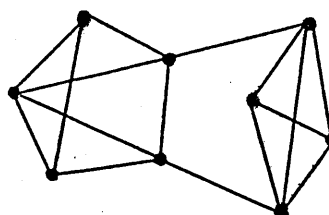


Figure 3.3

$J \neq K_5$ . Also, since  $G$  is 2-connected,  $J \neq K_5 \setminus e$ , or otherwise  $Y = X$  and  $|E(I)| = 5$ , contradicting the fact that  $\langle X \rangle = K_2 \cup 0_2$  or  $2K_2$ .

Suppose  $J = K_5 \setminus \{e_1, e_2\}$ , where  $e_1$  and  $e_2$  are independent. Then by examining the degree-list of  $J$ , we know that  $2 \leq m \leq 4$ , where  $m$  is the number of edges joining  $V(H)$  with  $Y$ , and  $4 \leq n = |E(I)| \leq 6$ . If  $m = 4$ , then all the vertices in  $V(H)$  are saturated, i.e. of degree 4, and thus  $Y = X$ , which is impossible because  $|E(\langle X \rangle)| \leq 2$  while  $|E(\langle Y \rangle)| = 4$ . If  $m = 3$ , then  $n = 5$ . It follows that  $Y$  has a vertex  $z$  of degree 4 joined to three vertices of degree 3 in  $Y$ , contradicting Lemma 2.1. If  $m = 2$ , then  $n = 6$  and  $G$  is one of the two graphs given in Figure 3.3. However, both these graphs are 4-colourable.

Suppose  $J = K_5 \setminus \{e_1, e_2\}$ , where  $e_1$  and  $e_2$  are not independent. Then again  $2 \leq m \leq 4$  and  $4 \leq n \leq 6$ . Similarly to the previous case, we can show that  $m \neq 4, 3$ . Suppose  $m = 2$ . Then  $n = 6$  and  $G$  is the graph given in Figure 3.4. However, this graph is also 4-colourable.

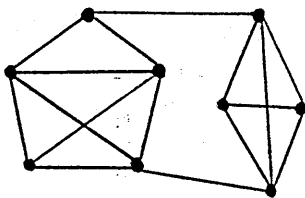


Figure 3.4

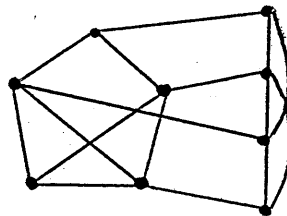


Figure 3.5

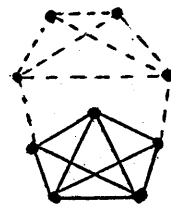


Figure 3.6

Finally, suppose  $J = H$ . Then there are four independent edges joining  $V(H)$  with  $Y$  because  $F \subseteq E(G) \setminus E(H)$ . Hence in this case  $4 \leq m \leq 6$  and  $3 \leq n \leq 5$ . If  $m = 6$ , then  $n = 3$  and  $Y = X$ , which is false. If  $m = 5$ , then  $n = 4$  and  $G$  has three vertices of degree 3 in  $Y$ . Let  $z \in Y$  be of degree 4. By Lemma 2.1,  $d_I(z) = 2$ ,  $I = \langle Y \rangle$  and  $\langle Y \setminus z \rangle = P_3$ , a path on three

vertices, contradicting the fact that  $\langle X \rangle = K_2 \cup 0_2$  or  $2K_2$ . If  $m = 4$ , then  $n = 5$  and  $G$  has two vertices of degree 3 in  $Y$ . In this case, it is clear that the four edges joining  $V(H)$  with  $Y$  constitute  $F$ . Hence it is not difficult to see that  $G$  must be the graph given in Figure 3.5.

However, this graph is 4-colourable.

The proof of Lemma 3.8 is now complete.

Lemma 3.9    Any 2-connected graph  $G$  having degree-list  $3^2 4^7$  is critical, except the graph of Figure 3.6.

Proof.        The size of  $G$  is  $17 > 4 \left\lfloor \frac{9}{2} \right\rfloor$ . Hence, by Lemma 2.7,  $G$  is Class 2. If  $G$  is not critical, then by Lemma 2.3  $G$  contains a 4-critical subgraph  $H$ . By the previous results, the order of  $H$  cannot be 9. Since there is no critical graph of even order  $\leq 10$ , the order of  $H$  is either 7 or 5.

If  $H$  is of order 7, then the degree-list of  $H$  is either  $24^6$  or  $3^2 4^5$  (see catalogue). But no graph  $H$  having degree-list  $24^6$  or  $3^2 4^5$  can be extended to a graph having degree-list  $3^2 4^7$ .

Next, there is only one 4-critical graph of order 5 and it can be extended in a unique way to a graph of order 9 having degree-list  $3^2 4^7$  as shown in Figure 3.6.

This completes the proof of Lemma 3.9.

Lemmas 3.5 - 3.9 together complete the proof of Theorem 3.1.

### 3.3 5-CRITICAL-GRAPHS

Lemma 3.10    Let  $G$  be a  $\Delta$ -critical graph of odd order  $n$  having size  $\geq \Delta \left( \frac{n-3}{2} \right) + \delta(G) + 1$ . Then for every  $x \in V(G)$  such that  $d(x) = \delta(G)$ ,  $G-x$  has a 1-factor  $F$ .

Proof. Let  $e_1$  be an edge incident with  $x$ . Let  $\pi$  be a  $\Delta$ -colouring of  $G \setminus e_1$  and let  $E_1, \dots, E_\Delta$  be the colour classes of  $E(G \setminus e_1)$ . Assume that  $|E_1| \geq \dots \geq |E_\Delta|$ . Since  $|E(G \setminus e_1)| \geq \Delta \left(\frac{n-3}{2}\right) + \delta$ , where  $\delta = \delta(G)$  and  $|E_1| = \dots = |E_\delta| = (n-1)/2$ . Now at least one of the colours  $1, \dots, \delta \notin C_\pi(x)$  because  $|C_\pi(x)| = d(x) - 1 = \delta - 1$ . The result follows by taking  $F = E_{i_1}$ , if  $i \notin C_\pi(x)$ ,  $i \leq \delta$ .

Open Problem. Is it true that if  $G$  is any  $\Delta$ -critical graph of odd order, then for each  $x \in V(G)$  such that  $d(x) = \delta(G)$ ,  $G \setminus x$  has a 1-factor  $F$ ?

Lemma 3.11. Let  $G$  be a 5-critical graph of order 9 and let  $x \in V(G)$  be such that  $d(x) = \delta(G)$ . Then  $G \setminus x$  has a 1-factor.

Proof. By Lemma 3.3 and Lemma 2.9,  $19 \leq |E(G)| \leq 21$ . Lemma 3.11 follows from Lemma 3.10 when  $|E(G)| \geq 20$  or when  $\delta(G) \leq 3$ . Now if  $\delta(G) = 4$ , then by Lemma 2.1,  $n_5 \geq 3$  and the degree-list of  $G$  is either  $4^5 5^4$  or  $4^3 5^6$ . Thus  $|E(G)| \geq 20$  and again Lemma 3.11 follows from Lemma 3.10.

Theorem 3.2 If  $G$  is a 5-critical graph of order 9, then  $|E(G)| = 21$ .

Proof. We shall prove that  $|E(G)| \neq 19$  or  $20$ . Suppose  $|E(G)| \leq 20$ . By Lemmas 3.11 and 3.4, for some vertex  $x$  of degree  $\delta(G)$ ,  $G \setminus x$  has a 1-factor  $F$  such that  $G \setminus F$  has a 4-critical subgraph  $H$ . By well-known results,  $|V(H)| \neq 8, 6$ . By the previous result on 4-critical graphs of order 9,  $|V(H)| \neq 9$ . Hence  $|V(H)| = 7$  or  $5$ .

Suppose  $|V(H)| = 7$ . Then the degree-list of  $H$  is either  $24^6$  or  $3^2 4^5$  (see catalogue) and  $|E(H)| = 13$ . Let  $\{v_1, v_2\} = V(G) \setminus V(H)$ .

Suppose  $v_1 v_2 \in E(G)$ . Then by Lemma 2.8,  $d(v_1) + d(v_2) \geq 7$ . Thus there are at least five edges joining  $\{v_1, v_2\}$  with  $V(H)$ . Moreover,  $F$  has at least two edges in  $\bar{H}$ . Hence  $|E(G)| \geq 13 + 2 + 5 + 1 = 21$ , contradicting our original assumption. Hence we may further assume that  $v_1 v_2 \notin E(G)$ .

It is clear that  $F$  has at least two but not more than three edges in  $\bar{H}$ . Suppose  $F$  has only two edges in  $\bar{H}$ . If  $|E(G)| = 19$ , then  $d(v_1) + d(v_2) = 19 - (13 + 2) = 4$  and so  $d(v_1) = 2 = d(v_2)$ ; if  $|E(G)| = 20$ ,  $d(v_1) + d(v_2) = 5$  and so  $d(v_1) = 2$  and  $d(v_2) = 3$ . In either case the only vertices of  $G$  which are of degree 2 are  $v_1$  and  $v_2$  and so in Lemma 3.11 we can choose  $x = v_1$ . But then  $F$  must have three edges in  $\bar{H}$ , contradicting our original assumption. This shows that  $F$  has three edges in  $\bar{H}$ . However, after adding three independent edges to  $H$ , the number of vertices of degree 5 in  $H \geq 4$ . Hence,  $v_1$  and  $v_2$  can only be joined to  $\leq 3$  vertices in  $H$  and so  $d(v_1) + d(v_2) \leq 3$  which is impossible.

Suppose  $|V(H)| = 5$ . Then  $H = K_5 \setminus \{u_1 u_2\}$ ,  $u_1, u_2 \in V(H)$  (see catalogue) and  $|E(H)| = 9$ . Let  $B$  be the set of edges joining  $V(H)$  with  $Z = V(G) \setminus V(H)$ .

Suppose  $u_1 u_2 \in E(G)$ . Then  $|B| \leq 5$ . Let  $I = \langle Z \rangle$ . If  $|B| = 5$ , then  $4 = 19 - (10 + 5) < |E(I)| < 20 - (10 + 5) = 5$ . Thus  $\delta(G) < \frac{1}{4}(5 + 2 \times 5)$  from which it follows that  $\delta(G) < 3$ . Hence  $F \subseteq G - x$  for some  $x \in Z$ . Now  $|Z \setminus x| = 3$  and  $|F| = 4$  imply that  $u_1 u_2 \in F$ . It is easy to show that  $d(x) \neq 2$ . Hence  $d(x) = 3$ . Since  $4 < |E(I)| \leq 5$ , there is  $z \in Z$  such that  $xz \in E(G)$ . Let  $Z = \{x, z, w_1, w_2\}$ . Then  $d(z) + d(w_1) + d(w_2) \leq 2 \times 5 + 5 - 3 = 12$ . Applying Lemma 2.1, it is not difficult to show that  $xw_1, xw_2 \in E(G)$  is impossible. Suppose  $xw_1 \in E(G)$  but  $xw_2 \notin E(G)$ . Then since  $z$  and  $w_1$  are both adjacent to at least 3 vertices of degree 5 and  $|B| = 5$ , we know that each of  $z$  and  $w_1$  is adjacent to 2 vertices in  $H$  and both  $z$  and  $w_1$  are adjacent to  $w_2$  where  $d(w_2) = 5$ . However, this is false because  $|B| = 5$ . Suppose  $xw_1, xw_2 \notin E(G)$ . Since  $F \subseteq G - x$ ,  $xu_1, xu_2 \in E(G)$  and each vertex in  $Z \setminus x$  is adjacent to exactly one vertex in  $H \setminus \{u_1, u_2\}$ . But then  $z$  cannot be adjacent to at least 3 vertices of degree 5. Hence  $|B| \neq 5$ . Now  $4 \geq |B| \geq 19 - (10 + 5) = 4$  implies that  $|B| = 4$ . Thus  $5 < |E(I)| \leq 6$ . In case  $|E(I)| = 6$ , applying Lemma 2.1 again, it is clear that  $d(x) \neq 3$ . Hence all the vertices in  $Z$  are of degree 4 and are adjacent to each other, contradicting Lemma 2.1. In case  $|E(I)| = 5$ ,  $\delta(G) < \frac{1}{4}(5 \times 2 + 4)$  from

which it follows that  $\delta(G) \leq 3$ . It is easy to show that  $\delta(G) \neq 2$ . Hence  $\delta(G) = 3$  and  $I = K_4 - xw_1$ , say, where  $w_1$  is of degree 3 and is adjacent to 2 vertices in  $Z$ , each of which is of degree at most 4, contradicting Lemma 2.1.

Suppose  $u_1u_2 \notin E(G)$ . Then  $F \subseteq B$ . Now  $F \subseteq G-x$  and  $F \subseteq B$  imply that  $x \in H$ . Hence  $\delta(G) \geq 3$  and  $|B| \leq 6$ . If  $\delta(G) = 3$ , we may assume that  $d(u_1) = 3$  and  $x = u_1$ . Then  $G$  is a subgraph of the graph given in Figure 3.7.

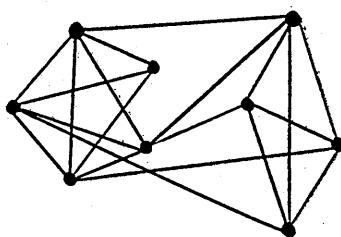


Figure 3.7

However, it is easy to verify that the graph in Figure 3.7 is 5-colourable. If  $\delta(G) = 4$ , then  $|B| \leq 6$  and  $|E(I)| = 4, 5$ , and 6. But if  $|E(I)| = 6$  then  $|B| \leq 5$ , which contradicts Lemma 2.1. Hence  $16 \leq \sum_{a \in I} d(a) \leq 5 \times 2 + 6$  showing that  $|B| = 6$ ,  $|E(I)| = 5$  and  $d(a) = 4$  for each  $a \in I$ . But then  $Z$  has a vertex  $b$  adjacent to a vertex  $c \in Z$  (having degree 4) and  $b$  is adjacent to only one vertex of degree 5, contradicting Lemma 2.1.

The proof of Theorem 3.2 is complete.

Corollary 3.1 A 2-connected graph of order 9 is 5-critical if and only if its degree-list is one of the following:  $25^8$ ,  $345^7$  or  $4^3 5^6$ .

Proof. By Theorem 3.2, if  $G$  is a 5-critical graph of order 9, then the degree-list is either  $25^8$ ,  $345^7$  or  $4^3 5^6$ .

To prove the converse, we first note that, since  $|E(G)| > 5 \left\lfloor \frac{9}{2} \right\rfloor$ ,  $G$  is Class 2. Hence if  $G$  is not critical, it contains a proper critical subgraph  $H$ . Again, by Theorem 3.2,  $|V(H)| \neq 9$ . Since there are no critical graphs of even order  $\leq 10$ ,  $|V(H)| = 7$  or  $5$ . Suppose  $|V(H)| = 7$ . Then the degree-list of  $G$  is either  $25^6$ ,  $345^5$  or  $4^3 5^4$  (see catalogue). However, none of the graphs having degree-lists  $25^6$ ,  $345^5$  and  $4^3 5^4$  can be extended to a graph having degree-list  $25^8$ ,  $345^7$  or  $4^3 5^6$ . It is also clear that  $|V(H)| \neq 5$ .

### 3.4 6-CRITICAL GRAPHS

Lemma 3.12 Let  $G$  be a  $\Delta$ -critical graph of odd order  $n$  having size  
 $\geq \Delta \left( \frac{n-3}{2} \right) + 2$ . Then  $G$  has a vertex  $y$  of degree  $< \Delta$  such that  $G-y$  has a  
1-factor  $F$ .

Proof. Let  $x \in G$  be such that  $d(x) = \delta(G)$  and let  $xz \in E(G)$  where  $d(z) = \Delta$ . Suppose  $\pi$  is a  $\Delta$ -colouring of  $G-xz$ . Let  $E_1, \dots, E_\Delta$  be the colour classes of  $E(G-xz)$  such that  $|E_1| \geq \dots \geq |E_\Delta|$ . Since  $|E(G-xz)| \geq \Delta \left( \frac{n-3}{2} \right) + 1$ ,  $|E_1| = (n-1)/2$ . Now if colour 1 is present at  $z$ , then  $F=E_1$  is a 1-factor of  $G-y$  for some  $y \in V(G)$  having degree  $< \Delta$ . Otherwise, by interchanging the colours 2 and 1 in the  $(2,1)_\pi$ -chain having initial vertex  $z$ , we reduce it to the previous case.

Corollary 3.2 Let  $G$  be a 6-critical graph of order 9. Then  $G$  has a  
vertex  $y$  of degree  $< \Delta$  such that  $G-y$  has a 1-factor  $F$ .

Proof. By Lemmas 3.3 and 2.9,  $21 \leq |E(G)| \leq 25$ . Corollary 3.2 now follows from Lemma 3.12.

Theorem 3.3 If  $G$  is a 6-critical graph of order 9, then  $|E(G)| = 25$ .

Proof. We shall show that  $|E(G)| \geq 25$ . Suppose  $|E(G)| \leq 24$ . By Corollary 3.2 and Lemma 3.4, there is  $y \in V(G)$  having degree  $< 6$ , so that  $G-y$  has a 1-factor  $F$  such that  $G-F$  is Class 2 and hence has a 5-critical



subgraph  $H$ . By now it should be obvious that  $|V(H)| \neq 6, 8$  or  $9$ . Hence  $|V(H)| = 7$ . The degree-list of  $H$  is either  $25^6$ ,  $345^5$  or  $4^3 5^4$  (see catalogue) and so  $|E(H)| = 16$ . Let  $\{v_1, v_2\} = V(G) \setminus V(H)$ . It is now also clear that  $v_1, v_2 \notin E(G)$  (see the third paragraph of the proof of Theorem 3.2).

Clearly  $F$  has at least two but not more than three edges in  $\bar{H}$ . Suppose  $F$  has only two edges in  $\bar{H}$ . Then  $d(v_1) + d(v_2) \leq |E(G)| - (16 + 2) = |E(G)| - 18$ . Let  $d(v_1) \leq d(v_2)$ . Hence if  $|E(G)| \leq 24$ ,  $d(v_1) \leq 3$ . We now show that for each  $w \in V(H)$ ,  $d(w) \geq 4$ .

Let  $H$  have degree-list  $345^5$ . Assume  $z \in V(H)$  is such that  $d_H(z) = 3$  and  $d(z) = 3$ . Then  $\sum_{w \in H} (d(w) - d_H(w)) - 2|E(\bar{H}) \cap F| \leq \sum_{w \in H \setminus z} (d(w) - d_H(w)) - 4 \leq 7 - 4 = 3$ , which is impossible. The case that the degree-list of  $H$  is  $25^6$  can be similarly disposed of.

The above shows that  $v_1$  and  $v_2$  are the only possible vertices of degree  $\delta(G)$ . Hence, by Lemma 3.10,  $G \setminus v_1$  has a 1-factor  $F$ , unless  $|E(G)| = 21$ . However, if  $|E(G)| = 21$ , then  $d(v_1) + d(v_2) \leq 21 - (16 + 2) = 3$ , which is not true. On the other hand, if  $F$  is a 1-factor of  $G \setminus v_1$ , then  $F$  has three edges in  $\bar{H}$ , contradicting our original assumption.

Finally, suppose  $F$  has three edges in  $\bar{H}$ . Then  $\sum_{w \in H} (d(w) - d_H(w)) - 2|E(\bar{H}) \cap F| \leq 4$  and so  $G$  has two vertices,  $v_1$  and  $v_2$ , of degree 2 and the remainder have degree 6, contradicting Lemma 3.1.

This completes the proof of Theorem 3.3.

Corollary 3.3. A 2-connected graph of order 9 is 6-critical if and only if its degree-list is one of:  $26^8$ ,  $356^7$ ,  $4^2 6^7$ ,  $45^2 6^6$  and  $5^4 6^5$ .

Proof. The proof is similar to that of Corollary 3.2.

### 3.5 7-CRITICAL GRAPHS

Lemma 3.13 Let  $G$  be a 7-critical graph of order 9. Then  $G$  has a vertex  $y$  having degree  $< \Delta$  such that  $G \setminus y$  has a 1-factor  $F$ .

Proof. By Lemma 3.3 and Lemma 2.9,  $23 < |E(G)| \leq 29$ . Lemma 3.13 now follows from Lemma 3.12.

Theorem 3.4. If  $G$  is a 7-critical graph of order 9, then  $|E(G)| = 29$ .

Proof. We shall prove that  $|E(G)| \neq 29$ . Suppose  $|E(G)| \leq 28$ . By Lemma 3.13 and 3.4,  $G$  has a near 1-factor  $F$  such that  $G \setminus F$  contains a 6-critical subgraph of  $H$ . The only possible order of  $H$  is 7. The degree-list of  $H$  is either  $45^2 6^4$  or  $5^4 6^3$ . Hence  $|E(H)| = 19$ . Let  $\{v_1, v_2\} = V(G) \setminus V(H)$ . It is now clear that  $v_1, v_2 \notin E(G)$  (See the third paragraph of the proof of Theorem 3.2) and  $d(v_1) \leq 3$ .

Since  $|E(H)| = 19$ ,  $F$  has exactly two edges in  $\bar{H}$ . Now if  $|E(G)| \geq 25$ , then by Lemma 3.10  $F$  is a 1-factor of  $G \setminus v_1$ , say, which is impossible.

Since  $|E(H)| = 19$ , using Lemma 3.1,  $F$  has exactly two edges in  $\bar{H}$ . Now if  $|E(G)| \geq 25$ , then by Lemma 3.10  $F$  could have been chosen to be a 1-factor of  $G \setminus v_1$ , which is impossible.

On the other hand, if  $|E(G)| \leq 24$ ,  $d(v_1) + d(v_2) \leq 24 - 21 = 3$  which is false.

The proof of Theorem 3.4 is complete.

Corollary 3.4. A 2-connected graph  $G$  of order 9 is 7-critical if and only if its degree-list is one of :  $27^8, 367^7, 457^7, 46^2 7^6, 5^2 67^6, 56^3 7^5$  and  $6^5 7^4$ .

Proof. This follows from Theorem 3.4 and the fact that there are no critical graphs of order 8.

### 3.6 8-CRITICAL GRAPHS.

Lemma 3.14 Let  $G$  be an 8-critical graph of order 9 and let  $x \in V(G)$  be of degree  $\delta(G)$ . Then  $G \setminus x$  has a 1-factor  $F$ .

Proof. If  $G' = G \setminus x$  does not have a 1-factor, then by Tutte's theorem (Lemma 2.13)  $V(G')$  has a subset  $S$  such that  $|S| < h = \text{number of odd components of } G' \setminus S$ . Now since  $|V(G')|$  is even,  $|S|$  and  $h$  must be of the same parity. Also since  $n_8 \geq 3$  (by Lemma 2.5),  $|S| \geq 3$  and  $h \geq 5$ . Hence  $|S| = 3$  and  $h = 5$ . However, this implies that  $G \subseteq K_4 + 0_5$ , the sum of  $K_4$  and  $0_5$ . Hence  $\delta(G) \leq 4$ . Applying Lemma 2.6, we have  $n_8 \geq 6$ , which contradicts the fact that  $|S| = 3$ .

Theorem 3.5 If  $G$  is an 8-critical graph of order 9, then  $|E(G)| = 33$ .

Proof. By Lemmas 3.14 and 3.4,  $G$  has a vertex  $x$  of degree  $\delta(G)$ , so that  $G \setminus x$  has a 1-factor  $F$  such that  $G \setminus F$  contains a 7-critical subgraph  $H$ . Now the order of  $H$  must be 9. However, by Theorem 3.4,  $|E(H)| = 29$  and thus  $|E(G)| \geq 29 + 4 = 33$ . Theorem 3.5 now follows from Lemma 2.9.

Remarks. Theorem 3.5 has been confirmed by a recent result of Plantholt; Plantholt's result is also discussed in Chapter 5, and follows from Theorem 4.1.

Corollary 3.5. A graph of order 9 is 8-critical if and only if its degree-list is one of:  $57^3 8^5$ ,  $6^3 8^6$ ,  $6^2 7^2 8^5$ ,  $67^4 8^4$  and  $7^6 8^3$ .

Proof. This follows from Theorem 3.5.

It is trivial that every critical graph is 2-connected. Also all the graphs with the degree lists mentioned in Theorems 3.1 - 3.5 are 2-connected. Therefore the hypothesis that  $G$  be 2-connected in these theorems can be dropped. However since graphs which are not connected are of no interest when considering criticality, it seems natural to retain the present statement.

## 4. Embedding Nearly Complete Graphs

### 4.1 Introduction

We now consider graphs  $G$  formed by removing  $n$  edges from the complete graph  $K_{2n+1}$ . We give a necessary and sufficient condition for any partial edge-colouring of  $G$  with  $2n$  colours to be extended to a proper edge-colouring of  $G$ .

In [B2] Beineke and Wilson observed that any graph  $G$  obtained by removing  $q$  edges ( $q < n$ ) from  $K_{2n+1}$  was Class 2 and therefore needed  $2n+1$  colours for an edge-colouring. In [H5] Hilton conjectured that if  $q = n$  then  $G$  would be Class 1 and therefore could be edge-coloured with  $2n$  colours. This would imply that if  $q = n-1$  then  $G$  would be a critical graph. Hilton's conjecture is a special case of an earlier conjecture of Vizing [V3] that, if  $G$  is critical, then

$$2|E(G)| \geq |V(G)|(|\Delta(G)|-1) + 3.$$

In Hilton's conjecture the inequality becomes an equality. In [P1], Plantholt proved Hilton's conjecture. In [A1, Corollary 4.3.3], Andersen and Hilton showed that an edge-colouring of  $K_r$  with  $2n-1$  colours can be extended to an edge-colouring of  $K_{2n}$  with the same colours if and only if each colour is used on at least  $r-n$  edges of  $K_r$ . In this chapter we show how an analogous result can be formulated and proved which implies Plantholt's theorem. Andersen and Hilton [A1] also showed that any partial edge-colouring of  $K_n$  with  $2n-1$  colours can be extended to a proper edge-colouring of  $K_{2n}$  with  $2n-1$  colours. We show that the situation can be very different if instead we consider  $K_{2n+1}$  with  $n$  edges removed and the colouring uses  $2n$  colours only.

We shall need the following result due to Hoffman and Kuhn [H6].

Lemma 4.1     A necessary and sufficient condition for a finite family  
 $(A_i : i \in I)$  of subsets of a finite set to have a system  $(x_i : i \in I)$  of  
distinct representatives such that  $M \subseteq \bigcup_{i \in I} x_i$  is that both of the  
following sets of inequalities are satisfied:

$$\text{I.} \quad \left| \bigcup_{i \in I'} A_i \right| \geq |I'| \quad (\forall I' \subseteq I),$$

$$\text{II.} \quad |\{i : A_i \cap M' \neq \emptyset \text{ and } i \in I\}| \geq |M'| \quad (\forall M' \subseteq M).$$

We shall call condition I, Hall's condition, and condition II, the marginal condition.

#### 4.2 Step-by-step extensions of edge-colourings.

Let  $G^*$  be a graph obtained by removing  $n$  edges from  $K_{2n+1}$ . Let the vertex set of  $G^*$  be  $\{v_1, v_2, \dots, v_{2n+1}\}$ . If  $d_{G^*}(v_1) \leq d_{G^*}(v_2) \leq \dots \leq d_{G^*}(v_{2n+1})$  then we say that  $G^*$  has a standard vertex labelling. With respect to a standard vertex labelling of  $G^*$ , for  $1 \leq r \leq 2n+1$ , let  $G_r^*$  be the restriction of  $G^*$  to  $\{v_1, \dots, v_r\}$ , or, in other words, the maximal induced subgraph of  $G^*$  with vertex set  $\{v_1, \dots, v_r\}$ ; let  $c_i = 2n - d_{G^*}(v_i)$ . Given an edge-colouring with a set of  $2n$  colours of  $G_r^*$  for some  $r$ ,  $1 \leq r \leq 2n+1$ , let  $e_r(\sigma)$  be the number of edges of  $G_r^*$  coloured with colour  $\sigma$ .

Theorem 4.1. Let  $1 \leq r \leq 2n+1$  and let  $G^*$  have a standard vertex labelling  
with respect to which  $G_r^*$  is defined.

A necessary and sufficient condition for an edge-colouring of  $G_r^*$   
with a set  $\mathcal{C}$  of  $2n$  colours to be extendible to an edge-colouring of  $G^*$   
with  $\mathcal{C}$  is that:

there are pairwise disjoint sets  $C_1, C_2, \dots, C_r$  of colours of  
 $\mathcal{C}$  such that

$$(i) \quad |C_i| = c_i \quad (1 \leq i \leq r),$$

$$(ii) \quad \text{no colour of } C_i \text{ is used in } G_r^* \text{ on an edge on the vertex } v_i \quad (1 \leq i \leq r),$$

and

$$(iii) \quad e_r(\sigma) \geq \begin{cases} r - n - 1 & \text{if } \sigma \in C_1 \cup \dots \cup C_r, \\ r - n & \text{otherwise.} \end{cases}$$

Proof.

Necessity. Suppose  $G^*$  has an edge-colouring with a set  $\mathcal{C}$  of  $2n$  colours. For  $1 \leq i \leq 2n+1$ , the vertex  $v_i$  has a set  $C_i$  of  $2n - d_{G^*}(v_i)$  colours missing from it. The number of edges of  $G^*$  is  $\binom{2n+1}{2} - n = (2n)n$ , so each colour is used on  $n$  edges exactly and is missing from exactly one vertex. Therefore each colour of  $C_i$ , the set of colours missing at  $v_i$ , is not missing in  $G^*$  at any other vertex. Therefore  $C_1, \dots, C_{2n+1}$  are pairwise disjoint, and so  $C_1, \dots, C_r$  are pairwise disjoint.

For  $\sigma \in \mathcal{C}$ , the number of edges of  $G^*$  coloured  $\sigma$  with at least one end on a vertex of  $\{v_{r+1}, \dots, v_{2n+1}\}$  is at most  $2n + 1 - r$ . Each colour is used in  $G^*$  on exactly  $n$  edges. Therefore, the number of edges coloured  $\sigma$  in  $G_r^*$  is at least  $n - (2n + 1 - r) = r - n - 1$ . On the other hand, if  $\sigma \in C_{r+1} \cup \dots \cup C_{2n+1}$ , then there is a vertex in  $\{v_{r+1}, \dots, v_{2n+1}\}$  at which  $\sigma$  does not appear, so the number of edges of  $G^*$  coloured  $\sigma$  with at least one end on a vertex of  $\{v_{r+1}, \dots, v_{2n+1}\}$  is at most  $2n - r$ . It follows that the number of edges coloured  $\sigma$  in  $G_r^*$  is at least  $n - (2n - r) = r - n$ .

Sufficiency. Let  $1 \leq r \leq 2n + 1$ , let  $\mathcal{C} = \{\sigma_1, \dots, \sigma_{2n}\}$  and let  $G_r^*$  satisfy conditions (i), (ii) and (iii). If  $r = 2n + 1$  then there is nothing to prove, so suppose  $r < 2n + 1$ . Let  $C_1 \cup \dots \cup C_r = \{\sigma_1, \dots, \sigma_{q_r}\}$ ; then  $q_r \geq r$ . We show that the given edge-colouring of  $G^*$  can be extended to

an edge-colouring of  $G_{r+1}^*$  satisfying (i), (ii), and (iii) (with  $r + 1$  replacing  $r$ , of course).

The description of the process we carry out is aided by the construction of the bipartite graph  $H$  we now describe. The vertex sets of  $H$  are  $\{v'_1, \dots, v'_r\} \cup \{w_1, \dots, w_{c_{r+1}}\}$  and  $\{\sigma'_1, \dots, \sigma'_{2n}, c^*\}$ . The edges of  $H$  are as follows. For  $1 \leq i \leq r$ ,  $1 \leq j \leq 2n$ ,  $v'_i$  is joined to  $\sigma'_j$  by an edge if  $\sigma'_j$  is not used on an edge on  $v_i$  and is not in the set  $C_i$ . An edge  $(v'_i, \sigma'_j)$  means that the colour  $\sigma'_j$  could be used on one of the edges from  $v_i$  to  $\{v_{r+1}, \dots, v_{2n+1}\}$ . The vertex  $v'_i$  is joined to  $c^*$  by  $2n + 1 - r - |\{\sigma'_j : \sigma'_j \text{ is joined to } v'_i\}|$  edges. This makes all vertices  $v'_i$  have degree  $2n + 1 - r$ . We now join the vertices  $w_1, \dots, w_{c_{r+1}}$  to all those colours which are not already used in  $C_1, C_2, \dots, C_r$ .

For  $c_r + 1 \leq j \leq 2n$  and  $1 \leq i \leq c_{r+1}$ ,  $\sigma'_j$  is joined to  $w_i$ .

The bipartite graph  $H$  is illustrated in Figure 4.1.

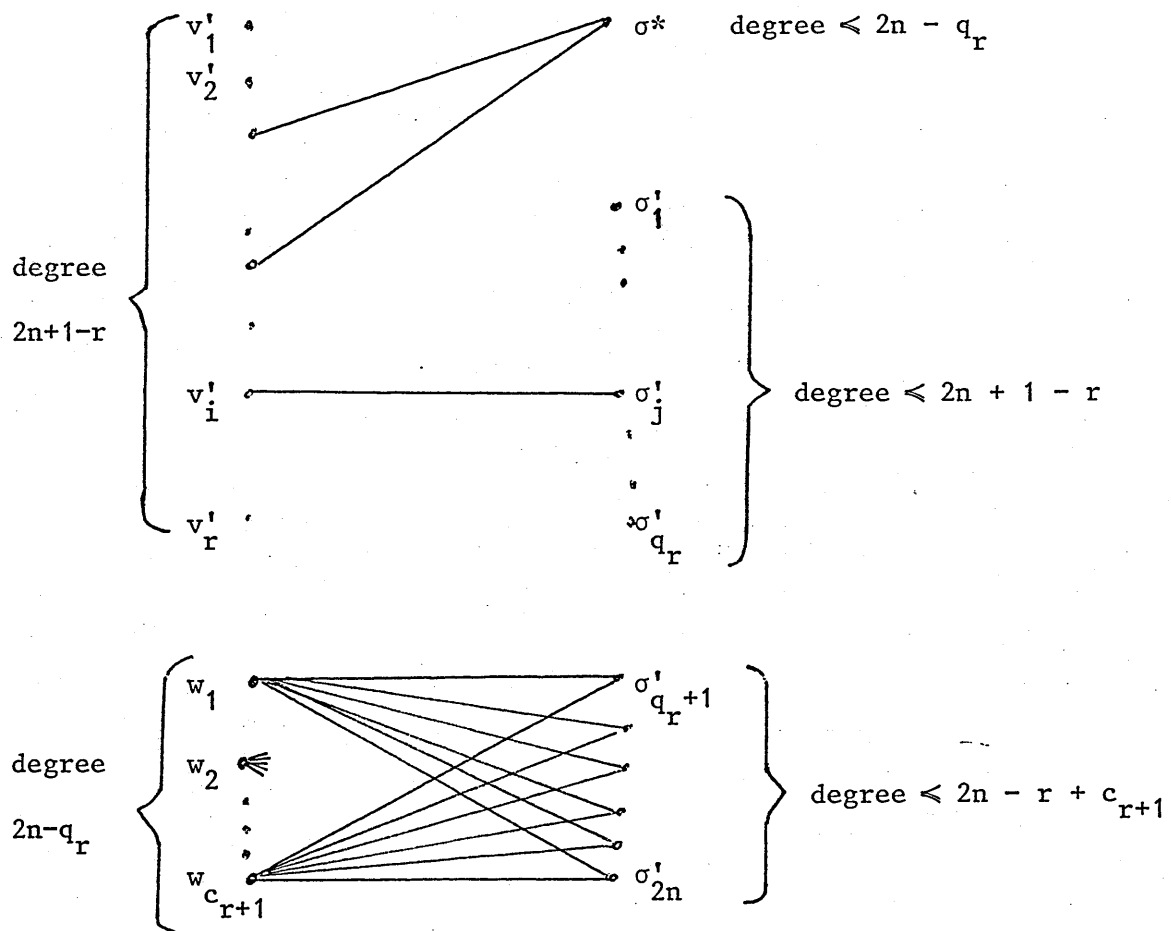


Figure 4.1 The graph H

Let  $H'$  be  $H \setminus \{\sigma^*\}$ . Then

$$\begin{aligned}
 d_H(v'_i) &= 2n + 1 - r & (1 \leq i \leq r), \\
 d_H(w_i) &= 2n - q_r & (1 \leq i \leq c_{r+1}), \\
 d_H(\sigma^*) &\leq 2n - q_r \\
 d_H(\sigma'_j) &\leq \begin{cases} 2n + 1 - r & (1 \leq j \leq q_r), \\ 2n - r + c_{r+1} & (q_r + 1 \leq j \leq 2n). \end{cases}
 \end{aligned}$$

The first and second equalities follow immediately from the definitions of  $\sigma^*$  and of  $w_1, \dots, w_{c_{r+1}}$  respectively. To show the inequality for  $d_H(\sigma^*)$ :



$$\begin{aligned}
d_H(\sigma^*) &= \sum_{i=1}^r \{(2n + 1 - r) - d_H(v_i^*)\} \\
&= \sum_{i=1}^r \{(2n + 1 - r) - (d_{G^*}(v_i) - d_{G_r^*}(v_i))\}
\end{aligned}$$

and so is the number of edges of  $\bar{G}^*$  which join vertices of  $\{v_1, \dots, v_r\}$  to vertices of  $\{v_{r+1}, \dots, v_{2n+1}\}$ . We know that the total deficiency is  $2n$ , and the deficiency of  $G_r^*$  is  $\sum_{i=1}^r c_i$ . Hence the deficiency of  $G^* \setminus G_r^*$  is

$$2n - \sum_{i=1}^r c_i = 2n - q_r,$$

and so the number of edges of  $\bar{G}^*$  from  $\{v_1, \dots, v_r\}$  to  $\{v_{r+1}, \dots, v_{2n+1}\}$  is less than or equal to this. Therefore  $d_H(\sigma^*) \leq 2n - q_r$ . Finally to show the inequality for  $d_H(\sigma_j^*)$ : If  $1 \leq j \leq q_r$  then, by assumption,  $e_r(\sigma_j) \geq r - n - 1$ . Therefore  $\sigma_j$  is not used on at most  $r - 2(r - n - 1) = 2n - r + 2$  vertices of  $G_r^*$ . However  $\sigma_j$  is in  $C_1 \cup \dots \cup C_r$ , so  $d_H(\sigma_j^*) \leq 2n - r + 1$ . If  $q_r + 1 \leq j \leq 2n$  then, by assumption,  $e_r(\sigma_j) \geq r - n$ . Therefore  $\sigma_j$  is not used on at most  $r - 2(r - n) = 2n - r$  vertices of  $G_r^*$ . However each vertex  $w_i$  ( $1 \leq i \leq c_{r+1}$ ) is joined to  $\sigma_j^*$  when  $q_r + 1 \leq j \leq 2n$ , so  $d_H(\sigma_j^*) \leq 2n - r + c_{r+1}$ .

Let  $M$  be the set of those  $\sigma_j$  such that

$$e_r(\sigma_j) = \begin{cases} r - n - 1 & \text{if } 1 \leq j \leq q_r, \\ r - n & \text{if } q_r + 1 \leq j \leq 2n. \end{cases}$$

These  $\sigma_j$  will be called marginal colours and the corresponding vertices  $\sigma_j^*$  will be called marginal vertices. We want to be sure that these marginal colours are used on the edges at  $v_{r+1}$  so that the number of times each colour is used satisfies the initial conditions.

We would like to find a set  $J$  of independent edges of  $H'$  which covers each vertex of  $\{v_1^*, \dots, v_r^*\} \cup \{w_1, \dots, w_{c_{r+1}}\}$  except for those  $v_i^*$

such that  $v_i$  is not joined in  $G_{r+1}^*$  to  $v_{r+1}$ , and which also covers each vertex of  $M$ . For then if  $v_i \sigma_j \in J$  then we colour  $v_i v_{r+1}$  with  $\sigma_j$ , and if  $w_i \sigma_j \in J$  then we place  $\sigma_j$  in the set  $C_{r+1}$ . Then  $c_{r+1}$  colours will be placed in  $C_{r+1}$ . Furthermore any marginal colour will either be used to colour an edge from  $v_{r+1}$  to one of  $\{v_1, \dots, v_r\}$ , or will be placed in  $C_{r+1}$ . By relabelling if necessary, we may assume that  $C_{r+1} = \{\sigma_{q_r+1}, \dots, \sigma_{q_{r+1}}\}$ . In the case where  $\sigma_j$  is used to colour an edge we have

$$e_{r+1}(\sigma_j) = \begin{cases} (r+1) - n - 1 & \text{if } 1 \leq j \leq q_r, \\ (r+1) - n & \text{if } q_{r+1} + 1 \leq j \leq 2n. \end{cases}$$

In the other case  $\sigma_j$  is used in  $C_{r+1}$  and so  $j \geq q_r + 1$ , but  $j \leq q_{r+1}$ : thus  $e_{r+1}(\sigma_j) = r - n = (r+1) - n - 1$  and  $q_r + 1 \leq j \leq q_{r+1}$ .

Thus marginal colours will satisfy (iii) with  $r+1$  replacing  $r$ , but will be marginal in that case also. Clearly non-marginal elements will satisfy (iii) with  $r+1$  replacing  $r$ .

Furthermore  $C_i \cap C_{r+1} = \emptyset$  ( $1 \leq i \leq r$ ), and it is easy to check that the remaining parts of the conditions (i), (ii) and (iii) will be satisfied with  $r+1$  replacing  $r$ . Thus it remains to demonstrate the existence of a suitable set  $J$ . We do this by verifying the Hoffman-Kuhn inequalities of Lemma 4.1.

We show first that Hall's condition is satisfied. Let  $W' \subseteq \{w_1, \dots, w_{c_{r+1}}\}$  and  $V' \subseteq \{v_1', \dots, v_r'\}$  and let  $V'$  contain no vertex  $v_i'$  such that  $v_i$  is not joined in  $G$  to  $v_{r+1}$ . Clearly if  $W' \neq \emptyset$  then  $|N_H'(W')| = 2n - q_r \geq |W'|$ . We consider various cases with  $V' \neq \emptyset$ .

Case H1.  $c_{r+1} \geq 3$  or ( $c_{r+1} = 2$  and  $c_{r+2} \neq 0$ ).

$$|N_{H'}(V' \cup W')|$$

$$> |N_{H'}(V')|$$

$$> |N_{H'}(v_i)| \text{ for some } v_i',$$

$$> 2n - c_i - d_{G_r}(v_i).$$

If  $2n - c_i - d_{G_r}(v_i) > r + c_{r+1}$  then we obtain

$$|N_{H'}(V' \cup W')| > |\{v_1, \dots, v_r\} \cup \{w_1, \dots, w_{c_{r+1}}\}| > |V'| + |W'|,$$

as required. If, however,  $2n - c_i - d_{G_r}(v_i) < r + c_{r+1}$  then

$$2n < r + c_i + c_{r+1} + d_{G_r}(v_i)$$

$$\text{and } d_{G_r}(v_i) < r - 1,$$

so

$$\sum_{j=1}^{2n} c_i = 2n < 2r + c_i + c_{r+1} - 1 < 2r + c_1 + c_{r+1} - 1$$

so

$$\sum_{j=2}^r c_i + \sum_{i=r+2}^{2n} c_i < 2r - 1.$$

Since  $c_{r+1} > 2$ , we know that  $c_i > 2$  for  $i < r + 1$ .

Hence  $|\sum_{j=2}^r c_i| > 2(r - 1)$  and it follows that  $\sum_{i=r+2}^{2n} c_i = 0$  and that  $c_2 = \dots = c_r = 2$ . Consequently  $c_{r+1} < 2$ , a case which is considered below.

Case H2.  $c_{r+1} = 2$  and  $c_{r+2} = \dots = c_{2n+1} = 0$ . Then

$$d_H(\sigma^*) < 2n - q_r = 2n - \sum_{i=1}^r c_i < 2n - (2n - 2) = 2.$$

Therefore

$$d_{H'}(v_i') > (2n + 1 - r) - 2 = 2n - r - 1 \quad (1 \leq i \leq r).$$

If  $2n - r - 1 \geq r + 2$  then for some  $i$ ,  $1 \leq i \leq r$ ,

$$|N_{H'}(v'_i)| \geq 2n - r - 1 \geq r + 2 = r + c_{r+1} \geq |V'| + |W'|,$$

from which Hall's condition follows. On the other hand, if  $2n - r - 1 < r + 2$  then  $2n < 2r + 3$ , so  $n - \frac{3}{2} < r$ . But  $2(r + 1) \leq \sum_{i=1}^{r+1} c_i = 2n$ , so

$r \leq n - 1$ . Therefore, in this case,  $r = n - 1$  and  $c_1 = \dots = c_{r+1} = 2$ .

It follows that  $d_{H'}(v'_i) \geq 2n - r - 1$ . If  $d_{H'}(v'_i) = 2n - r - 1$  then this implies that  $v_i$  is not joined to 2 elements of  $\{v_{r+1}, \dots, v_{2n+1}\}$ , but this is not possible, since  $c_{r+1} = 2$  and  $c_{r+i} = 0$ ,  $i = 2, \dots, 2n - r$ . Therefore  $d_{H'}(v'_i) = 2n - r$  or  $2n - r + 1$  ( $1 \leq i \leq r$ ). Therefore for any  $i$ ,  $1 \leq i \leq r$ ,

$$|N_{H'}(v'_i)| \geq 2n - r = n + 1 = r + 2 \geq |V'| + |W'|,$$

from which Hall's condition again follows.

Case H3.  $c_{r+1} \leq 1$ . Then  $|W| \leq 1$ . Consider the minimal subgraph  $H''$  of  $H$  containing the set  $V' \cup W'$  of vertices and all edges of  $H$  on these vertices. The number of edges in it is

$$\begin{cases} |V'| (2n + 1 - r) & \text{if } |W'| = 0, \\ |V'| (2n + 1 - r) + (2n - q_r) & \text{if } |W'| = 1. \end{cases}$$

Each vertex  $\sigma'_i$  has degree in  $H$  (and therefore in  $H''$ ) at most  $2n + 1 - r$ , and  $\sigma^*$  has degree in  $H$  at most  $2n - q_r$ . It follows that if  $d_{H''}(\sigma^*) < 2n - q_r$  then the number of  $\sigma$ -vertices in  $H''$  is at least  $|V'| + |W'| + 1$ , so

$$(1) \quad |N_{H''}(V')| + |N_{H''}(W')| \geq |V'| + |W'|.$$

If  $d_{H''}(\sigma^*) = 2n - q_r = 0$ , then (1) is similarly true. If

$d_{H''}(\sigma^*) = 2n - q_r > 0$ , then we know that in  $\bar{G}$  all edges on vertices  $\{v_{r+1}, \dots, v_{2n+1}\}$  have their other vertex in  $\{v_1, \dots, v_r\}$ . Now since  $2n - q_r \neq 0$ , we know that  $c_{r+1} \geq 1$ , and since  $c_{r+1} \leq 1$  we have  $c_{r+1} = 1$ , so there is a vertex  $v_i$  in  $\{v_1, \dots, v_r\}$  not joined to  $v_{r+1}$ . So by the definition of  $V'$ ,  $v_i \notin V'$ . But this contradicts  $d_{H''}(\sigma^*) = 2n - q_r$ .

Thus, in all cases, Hall's condition is satisfied.

Next we show that the marginal condition is satisfied. First we note that if

$$e_r(\sigma) > \begin{cases} r - n - 1 & \text{for } \sigma \in C_1 \cup \dots \cup C_r, \\ r - n & \text{otherwise,} \end{cases}$$

then  $\sigma$  is not marginal. Therefore, if  $n > r$  there are no marginal symbols, so that the marginal condition is satisfied vacuously.

Suppose therefore from now that  $r \geq n$ . Then it follows that  $c_{r+1} \leq 1$ . For if  $c_{r+1} \geq 2$  we would obtain the following contradiction:

$$2n = \sum_{i=1}^{2n+1} c_i \geq \sum_{i=1}^{r+1} c_i \geq 2(r+1) > 2r \geq 2n.$$

Thus  $c_{r+1} \leq 1$ . Therefore there is at most one vertex, say  $v_r^+$ , such that  $v_r^+ \in \{v_1, \dots, v_r\}$  and  $v_r^+$  is not joined in  $G$  to  $v_{r+1}$ .

Let  $M'$  be a set of marginal elements  $\sigma_i^+$ . We wish to show that

$$|N_{H'}(M')| \geq |M'| \text{ if } v_r^+ \text{ does not exist,}$$

$$|N_{H'}(M') \setminus \{v_r^+\}| \geq |M'| \text{ otherwise.}$$

Consider the subgraph of  $H'$  consisting of the vertices of  $M'$  and the vertices of

$$\begin{cases} N_{H'}(M') & \text{if } v_r^+ \text{ does not exist,} \\ N_{H'}(M') \setminus v_r^+ & \text{otherwise,} \end{cases}$$

and all edges of  $H'$  between these vertices. The number of such edges is equal to

$$|M'| d(\sigma') \text{ if } v_r^+ \text{ does not exist,}$$

and at least

$$|M'| d(\sigma') - d(v_r^+) \text{ otherwise.}$$

For  $\sigma_i \in M$  we know that

$$e_r(\sigma_i) = \begin{cases} r - n - 1 & \text{for } \sigma_i \in (C_1 \cup \dots \cup C_r) \cap M, \\ r - n & \text{for } \sigma_i \in M \setminus (C_1 \cup \dots \cup C_r). \end{cases}$$

$$\text{Hence } d(\sigma_i) = \begin{cases} n - (r - n - 1) & \text{for } \sigma_i \in (C_1 \cup \dots \cup C_r) \cap M, \\ n - (r - n) + 1 & \text{for } \sigma_i \in M \setminus (C_1 \cup \dots \cup C_r), \end{cases}$$

and so

$$d(\sigma_i) = 2n - r + 1 \quad (\forall \sigma_i \in M).$$

Therefore the number of edges in  $H'$  from  $M'$  is equal to

$$= |M'| (2n - r + 1) \quad \text{if } v^+ \text{ does not exist}$$

and is at least

$$\begin{aligned} |M'| (2n - r + 1) - d(v^+) &\geq |M'| (2n - r + 1) - (2n - r) \\ &= (|M'| - 1) (2n - r + 1) + 1 \quad \text{otherwise.} \end{aligned}$$

The marginal condition now follows from the fact that the maximum degree in the subgraph is  $(2n - r + 1)$ .

This proves the sufficiency and completes the proof of Theorem 4.1.

#### 4.3. Extending partial edge-colourings.

A partial edge-colouring of a graph  $G$  is an edge-colouring of some subgraph  $G'$  of  $G$ . Andersen and Hilton showed in [A1] that any partial edge-colouring of  $K_n$  with  $(\leq 2n-1)$  colours can be extended to an edge-colouring of  $K_{2n}$  with  $2n-1$  colours. We investigate the analogous problem for partial edge-colourings of  $G_r^*$ .

We first of all give a necessary and sufficient condition for the extendibility of an edge-colouring of  $G_r^*$ , where  $r \leq n + 1$ ; we only have to consider the sets of colours unused at each vertex.

Theorem 4.2. Let  $r \leq n + 1$  and let  $G^*$  have a standard vertex labelling.

Let  $G_r^*$  be edge-coloured with  $2n$  colours, and, for  $1 \leq i \leq r$ , let  $D_i$  be the set of colours not used at vertex  $v_i$ . This edge colouring of  $G_r^*$  can be extended to an edge-colouring of  $G^*$  with  $2n$  colours if and only if

$$\left| \bigcup_{i \in I} D_i \right| > \sum_{i \in I} c_i \quad (\forall I \subseteq \{1, \dots, r\}).$$

Proof. If we assume that  $G_r^*$  can be extended, then by Theorem 4.1 we know that there are pairwise disjoint sets  $C_1, \dots, C_r$  such that  $|C_i| = c_i$  and no colour of  $C_i$  is used in  $G_r^*$  on an edge at  $v_i$ . Hence  $C_i \subseteq D_i$  and

$$\left| \bigcup_{i \in I} D_i \right| > \left| \bigcup_{i \in I} C_i \right| = \sum_{i \in I} c_i \quad (\forall I \subseteq \{1, \dots, r\}).$$

We now assume that

$$\left| \bigcup_{i \in I} D_i \right| > \sum_{i \in I} c_i \quad (\forall I \subseteq \{1, \dots, r\})$$

and show that we can find pairwise disjoint sets of colours  $C_i$  ( $i = 1, \dots, r$ ) such that (i), (ii) and (iii) of Theorem 4.1 are satisfied.

To find the  $C_i$  it is sufficient to show that the family

$$(D_{11}, \dots, D_{1c_1}, D_{21}, \dots, D_{2c_2}, \dots, D_r, \dots, D_{rc_r}),$$

where, for  $1 \leq i \leq r$ ,  $D_{i1} = \dots = D_{ic_i} = D_i$ , has a system of distinct representatives. For then, by Hall's theorem [H6],

$$\left| \bigcup_{i \in I} \bigcup_{j \in J_i} D_{ij} \right| > \sum_{i \in I} |J_i|$$

$$(\forall I \subseteq \{1, \dots, r\}, i \in I, J_i \subseteq \{1, \dots, c_i\}).$$

But this is equivalent to saying that

$$\left| \bigcup_{i \in I} D_i \right| > \sum_{i \in I} c_i \quad (\forall I \subseteq \{1, \dots, r\}),$$

which is true by assumption.

Condition (i) holds since  $|C_i| = c_i$  and (ii) is true since  $C_i \subseteq D_i$ .

We now show that (iii) holds. If  $r \leq n$  then this condition is vacuous, so assume  $r = n+1$ . Suppose that there is a colour  $\sigma$  which is not in

$C_1 \cup \dots \cup C_{n+1}$  and that  $\sigma$  does not occur on any edge of  $G_{n+1}^*$ .  $G_{n+1}^*$  has a vertex of degree at least  $n-1$ , for otherwise  $\sum_{i=1}^{n+1} c_i > 2(n+1)$ , which

contradicts the fact that  $\sum_{i=1}^n c_i = 2n$ . So there are at least  $n-1$  colours used in  $G_{n+1}^*$ . One of these colours, say  $t$ , must be in  $C_1 \cup \dots \cup C_r$  since  $|C_1 \cup \dots \cup C_r| \geq n+1$ . We replace the colour  $t$  in  $C_1 \cup \dots \cup C_r$  by the colour  $\sigma$ . We can repeat this operation if necessary. This shows that (iii) can be made true consistently with (i) and (ii), and hence that  $G_r^*$  can be extended to an edge-colouring of  $G^*$ .

Corollary 4.1. Let  $r \leq n + 1$ . If  $2n - r + 1 \geq c_1 + \dots + c_r$  then any partial edge-colouring of  $G_r^*$  with  $2n$  colours can be extended to an edge-colouring of  $G^*$ .

Proof. The maximum degree in  $G_r^*$  of any vertex  $v_i$  of  $G_r^*$  is at most  $r-1$  and hence  $|D_i| \geq 2n-r+1$  ( $1 \leq i \leq r$ ). Therefore

$$|\bigcup_{i \in I} D_i| \geq 2n - r + 1 \geq \sum_{i \in I} c_i \quad (\forall I \subseteq \{1, \dots, r\}),$$

so the corollary follows from Theorem 4.2.

We now give a theorem which shows that it is possible to obtain graphs  $G^*$  and partial subgraphs  $G_n^*$  where an edge-colouring of  $G_n^*$  is not extendible to  $G^*$ . This is in sharp contrast to the result that a partial edge-colouring of  $K_n$  with  $2n-1$  colours can always be extended to an edge-colouring of  $K_{2n}$  with  $2n-1$  colours.

Theorem 4.3. Let  $r$  be even and  $\binom{\frac{r}{2}}{2} \geq n \geq \frac{r}{2}$ . Then there are graphs  $G^*$  and partial edge colourings of  $G_r^*$  with  $2n$  colours such that the partial edge colouring of  $G_r^*$  cannot be extended to an edge-colouring of  $G$ .

Proof. Let  $G^*$  be the graph obtained by removing from  $K_{2n+1}$ ,  $n$  edges whose end vertices all lie in a set  $\{v_1, \dots, v_{\frac{r}{2}}\}$  of  $\frac{r}{2}$  vertices.

This is possible since  $\binom{\frac{r}{2}}{2} \geq n$  and  $\frac{r}{2} \leq 2n+1$ . Let  $F$  be a set of  $\frac{r}{2}$  independent edges of  $G_r^*$ , each edge having exactly one



end in the set  $\{v_1, \dots, v_{\frac{r}{2}}\}$ , and colour  $G_r^*$  with  $(\leq 2n)$  colours so that all the edges of  $F$  receive the same colour.

To show that we can colour  $G_r^*$  in this way, consider  $G_r^* \setminus F$ . This has maximum degree  $r-2$  and hence can be edge-coloured with  $r-1$  colours. Colour  $F$  with a new colour. Then  $G_r^*$  is edge-coloured with  $r$  colours (and  $r \leq 2n$ ). Then, since one colour is used at every vertex,

$$\left| \bigcup_{i \in \{1, \dots, r\}} D_i \right| \leq 2n-1$$

whereas

$$\sum_{i \in \{1, \dots, r\}} c_i = 2n.$$

Therefore by Theorem 4.2, the edge-colouring cannot be extended to an edge-colouring of  $G^*$  with  $2n$  colours.

This proves Theorem 4.3.

In contrast to this, we show in the next theorem that if  $c_1 = n$  then any edge-colouring of  $G_{n+1}^*$  can be extended to an edge-colouring of  $G^*$  with  $2n$  colours.

Theorem 4.4. If  $c_1 = n$  then any edge-colouring of  $G_{n+1}^*$  with  $2n$  colours can be extended to an edge-colouring of  $G^*$  with  $2n$  colours.

Proof. Since  $c_1 = n$ , it follows that  $c_2 = c_3 = \dots = c_{n+1} = 1$  and that in  $G_{n+1}^*$ ,  $v_1$  is not joined to any other vertex. Thus  $|D_1| = 2n$ . Since  $|D_i| \geq n+1$  ( $2 \leq i \leq n+1$ ),

$$\begin{aligned} \text{if } 1 \in I \text{ then } \bigcup_{i \in I} D_i &\geq 2n > \sum_{i=1}^{n+1} c_i > \sum_{i \in I} c_i, \text{ and} \\ \text{if } 1 \notin I \text{ then } \bigcup_{i \in I} D_i &\geq n+1 > \sum_{i=2}^{n+1} c_i > \sum_{i \in I} c_i, \end{aligned}$$

so Theorem 4.4 follows from Theorem 4.2.

Given  $r$ , let  $n_0$  be the least integer such that if  $n \geq n_0$  then, for any graph  $G^*$  formed from  $K_{2n+1}$  by removing  $n$  edges, any edge-colouring of  $G_r^*$  with  $2n$  colours can be extended to an edge-colouring of  $G$  with  $2n$  colours. Theorem 4.3 shows that  $n_0 \geq \binom{\frac{r}{2}}{2} + 1$ . We make the following conjecture.

Conjecture 4.1.

$$n_0 = \binom{\frac{r}{2}}{2} + 1 .$$

## 5. Nearly complete graphs of odd order.

### 5.1 Introduction

In [P1] Plantholt showed that if we remove  $q$  edges from  $K_{2n+1}$  then the resulting graph  $G$  has  $\chi'(G) = 2n+1$  if and only if  $q < n$ . We have already proved this by an inductive argument in Chapter 4. We now give another proof of Plantholt's theorem and also give the "next case" of Vizing's conjecture (see Section 4.1), namely that if  $2n$  edges are removed from  $K_{2n+1}$ , giving a graph  $H$ , and if the maximum degree of  $H$  is  $2n-1$ , then  $H$  is edge-colourable with  $2n-1$  colours.

We shall make use of the following result. An edge-colouring of a graph is equalized if  $||E_i| - |E_j|| \leq 1$ , whenever  $E_i$  and  $E_j$  are the sets of edges of  $G$  of two distinct colours. McDiarmid [M1] and de Werra [W2] proved the following.

Lemma 5.1 If  $G$  has an edge-colouring with a set of colours, then it has an equalized edge-colouring with the same set of colours.

### 5.2 The second case of Vizing's Conjecture

First we need the following lemma.

Lemma 5.2 Let  $G$  be a multigraph with at most two vertices  $b$  (and possibly  $c$ ) of highest degree, let all the non-simple edges be incident with  $b$ , and if  $b$  and  $c$  are joined by more than one edge, let there be a vertex  $w$  such that  $w$  is joined to  $c$  but not to  $b$ . Let  $G$  not contain a subgraph on three vertices with  $\Delta(G) + 1$  edges. Then

$$\chi'(G) = \Delta(G).$$

Proof. Let  $W$  be the set of vertices of  $G$  joined in  $G$  to  $b$  by non-simple edges. Let  $H$  and  $H^*$  be the simple subgraph and the sub-multigraph respectively of  $G$  induced by  $W \cup \{b\}$ . We show first that  $\chi'(H^*) \leq \Delta(G)$ .

If  $H$  is Class 1, then edge-colour  $H$  with  $|V(H)| - 1$  colours, and extend this to an edge-colouring of  $H^*$  by colouring each of the extra edges on  $b$  with an extra colour. Then the number of colours used will be  $d_{H^*}(b) \leq \Delta(G)$ .

If  $H$  is Class 2, then edge-colour  $H$  with  $|V(H)|$  colours. If the colour, say  $\alpha$ , missing at  $b$  is also missing at some other vertex, say  $v^*$ , then colour one of the extra edges joining  $b$  to  $v^*$  with  $\alpha$ , and colour the remaining extra edges on  $b$  with an extra colour. Then the number of colours used will be  $d_{H^*}(b) \leq \Delta(G)$ . On the other hand, if  $\alpha$  is missing at no other vertex, then  $|V(H)|$  is odd. Colour the extra edges on  $b$  with extra colours. Then the total number of colours used will be  $d_{H^*}(b) + 1$ . If  $d_{H^*}(b) + 1 \leq \Delta(G)$  then this is the desired edge-colouring of  $H^*$ . Since  $d_{H^*}(b) \leq \Delta(G)$ , we have used at most  $\Delta(G) + 1$  colours, or at most one colour too many. In that case we replace the colour  $\alpha$  on all the  $\frac{1}{2}(|V(H)| - 1)$  edges of  $H$  at which it now occurs by extra colours. Provided  $|V(H)| \geq 5$ , there is such an extra colour which is not present in  $H^*$  on either of the vertices of an edge coloured  $\alpha$ . Thus in this case, if  $|V(H)| \geq 5$ , then  $\chi'(H^*) \leq \Delta(G)$ . If  $|V(H)| = 3$ , since  $G$  does not contain a subgraph on three vertices with  $\Delta(G) + 1$  edges, the number of edges in  $H^*$  is at most  $\Delta(G)$ , so clearly  $\chi'(H^*) \leq \Delta(G)$ .

We now show how to obtain from this an edge-colouring of  $G$  with  $\Delta(G)$  colours. First colour  $E(G \setminus H^*)$  with  $\Delta(G)$  colours. This can be done by Lemma 2.5, since  $G \setminus H^*$  has at most one vertex of degree  $\Delta(G)$ . Then colour the edges of  $G$  joining  $H^*$  to  $G \setminus H^*$  using Vizing's original argument [V1], always having the pivot vertex of each of the fans in  $V(G \setminus H^*)$ , and choosing the last edge, or the last two edges, as follows. If  $b$  is the only vertex of maximum degree, choose an edge on  $b$  last. If  $b$  and  $c$  both exist, suppose first that  $c \in V(G \setminus H^*)$ . If  $c$  is joined to  $b$ , then colour the edge  $bc$  last. If  $c$  is not joined to  $b$ , colour an edge on

b next to last, and an edge on c last. If  $c \in V(H^*)$  then colour an edge on b next to last and the edge wc last. Vizing's original argument will apply as, at any stage during the construction of the fans, there will always be a further colour available on the vertex at the other end from the pivot of the most recently adjoined edge; except at the final stage, such a further colour is then used to define the next edge of the fan. This yields the desired colouring of  $G$  with  $\Delta(G)$  colours.

We have already found a direct proof of Plantholt's theorem using Theorem 4.1. We next give another way of proving Plantholt's theorem, also using Theorem 4.1; the amount by which the proof depends on Theorem 4.1 varies (according to the choice of  $r$ ).

Proof of Plantholt's theorem. Let  $1 \leq r \leq 2n$ . From  $G_r^*$  construct a multigraph  $G_r^{**}$  by adjoining a further vertex  $v^*$  and, for  $1 \leq i \leq r$ ,  $c_i$  edges joining  $v^*$  to  $v_i$ . Since  $v_i$  is joined to  $v_{2n+1}$  we know that  $d(v_i)$  in  $G_r^{**}$  is  $\leq 2n-1$  ( $\forall i, 1 \leq i \leq r$ ). By Lemma 5.2,  $\chi'(G_r^{**}) \leq \sum_{1 \leq i \leq r} c_i \leq 2n$ . By Lemma 5.1, an edge-colouring of  $G_r^{**}$  can be equalized; then the conditions of Theorem 4.1 are satisfied by  $G_r^*$ , so that  $G^*$  can be edge-coloured with  $2n$  colours.

When  $r = 2n$ , Theorem 4.1 is so trivial that the above proof of Plantholt's theorem is really self-contained. The proof of the next theorem, the 'next-case' of Vizing's conjecture, is essentially an imitation of the " $r = 2n$  proof" above of Plantholt's theorem.

Theorem 5.1 Let  $H$  be a simple graph such that  $|V(H)| = 2n + 1$ ,  $|E(H)| = \binom{2n+1}{2} - 2n$  and  $\Delta(H) = 2n - 1$ . Then  $\chi'(H) = 2n-1$ .

Proof. First suppose that  $n \geq 4$ . There is in  $H$  a vertex, say  $v_{2n+1}$ , of degree  $2n - 1$ . Let the other vertices be  $v_1, \dots, v_{2n}$ . Let  $H' = H - v_{2n+1}$ . Then  $H'$  has one vertex, say  $v_{2n}$ , of degree at most  $2n - 1$ , the remainder having degree at most  $2n - 2$ . For  $1 \leq i \leq 2n + 1$ , let  $h_i = (2n-1) - d_H(v_i)$ .

Let  $H'^*$  be formed from  $H'$  by adjoining a further vertex  $v^*$  and inserting  $h_i$  edges between  $v^*$  and  $v_i$ , for  $1 \leq i \leq 2n$ . Then

$$d_{H'^*}(v^*) = \sum_{1 \leq i \leq 2n} h_i = 2n - 1.$$

Also  $d_{H'^*}(v_{2n}) = 2n - 1$  and  $d_{H'^*}(v_i) = 2n - 2$  ( $1 \leq i \leq 2n - 1$ ). Except in the case when  $v_{2n}$  and  $v^*$  are joined by more than one edge and there does not exist a vertex  $w$  joined to  $v_{2n}$  and not to  $v^*$ , then, by Lemma 5.2,  $H'^*$  has an edge-colouring with  $2n-1$  colours. In the exceptional case, since every vertex to  $v_{2n}$  is also joined to  $v^*$ , there can be no further multiple edges. We remove an independent set  $F$  of  $n$  edges containing an edge on  $v_{2n}$  and an edge on  $v^*$  (but not an edge joining  $v_{2n}$  to  $v^*$ ) and avoiding a vertex  $v'$  which was not joined to either of  $v_{2n}$  or  $v^*$ . Since  $n \geq 4$ , it is easy to see that there is such an  $F$ , as follows. Consider  $H'^* \setminus \{v_x, v^*, v'\}$  where  $v_x$  is a vertex other than  $v_{2n}$  joined to  $v^*$ . This graph has  $2n-2$  vertices and minimum degree  $2n-5$ . Now by Dirac's theorem [D2] there exists a Hamilton cycle if  $2n-5 \geq \frac{1}{2}(2n-2)$ , i.e.  $n \geq 4$ . We take alternate edges of the cycle together with edge  $v_x v^*$  for  $F$ . Let  $e$  be an edge on  $v'$ . By Lemma 5.2, we can now colour  $H'^* \setminus (F \cup e)$  with  $2n-2$  colours. Then colour  $e$  also, using Vizing's argument with  $v'$  as the pivot vertex. Then again we obtain an edge-colouring of  $H'^*$  with  $2n-1$  colours. In both cases, in the edge-colouring obtained,  $v_1, \dots, v_{2n-1}$  each have one colour missing, since the number of edges of  $H'^*$  is  $n(2n-1)$  and every colour is on at most  $n$  edges. Therefore each colour is on exactly  $n$  edges and each colour is missing from exactly one vertex. Therefore  $v_{2n+1}$  can be adjoined to  $H'^*$  and, for  $1 \leq i \leq 2n - 1$ , the edge  $v_i v_{2n+1}$  inserted with the colour on it being the colour missing at  $v_i$  in  $H'^*$ . Finally  $v^*$  can be deleted. This yields the graph  $H$  edge-coloured with  $2n - 1$  colours.

For  $n \leq 3$ , the theorem is easily deduced from results of Fiorini [F4] and Chapter 3.

This proves Theorem 5.1.

From Plantholt's theorem and Theorem 5.1 one may easily deduce that the edge-chromatic class which a graph on  $2n + 1$  vertices with at least  $\binom{2n+1}{2} - (3n - 1)$  edges belongs to is determined solely by the maximum degree and the number of edges, as indicated in Chart 1.

Number of edges	Class
$\binom{2n+1}{2}$	2
$\vdots$	
$\binom{2n+1}{2} - (n - 1)$	1
$\binom{2n+1}{2} - n$	
$\binom{2n+1}{2} - (n + 1)$	2 if $\Delta(G) = 2n - 1$ 1 if $\Delta(G) = 2n$
$\vdots$	
$\binom{2n+1}{2} - (2n - 1)$	
$\binom{2n+1}{2} - 2n$	1
$\binom{2n+1}{2} - (2n + 1)$	2 if $\Delta(G) = 2n - 2$ 1 if $\Delta(G) \geq 2n - 1$
$\vdots$	
$\binom{2n+1}{2} - (3n - 1)$	

Chart 1

### 5.3. Some final remarks and conjectures

Plantholt's Theorem and Theorem 5.1 verify the case  $r = 1$  and  $r = 2$  of the following conjecture.

Conjecture 5.1 Let  $1 \leq r \leq n$ . Let  $G$  be a simple graph with  $2n+1$  vertices and maximum degree  $\Delta(G) = 2n+1-r$ . Then  $G$  is Class 2 if and only if ,  
for some  $s$  such that  $0 \leq s \leq \frac{r-1}{2}$  and for some set  $\{v_1, v_2, \dots, v_{2s}\} \subseteq V(G)$ ,  

$$|E(G \setminus \{v_1, v_2, \dots, v_{2s}\})| > \binom{2n+1-2s}{2} - (r-2s)(n-s).$$

For  $1 \leq s \leq \frac{r-1}{2}$ , we can construct examples of graphs of Class 2 which do not satisfy the inequality with  $1 \leq s \leq \sigma$  but do satisfy it for  $s = \sigma$ . Let  $H$  be a graph obtained from a  $K_{2(n-s)+1}$  by removing  $(r-2s)(n-s)-1$  edges in such a way that the maximum degree  $\Delta(H)$  of  $H$  is given by  $\Delta(H) = 2(n-s) + 1 - (r - 2s) = 2n-r+1$ . The graph  $H^*$  consisting of  $H$  together with  $2s$  isolated vertices is such an example. Then  $H^*$  is Class 2 since

$$\begin{aligned} |E(H^*)| &= \binom{2(n-s)+1}{2} - [(r-2s)(n-s)-1] \\ &= [2(n-s)+1 - (r-2s)](n-s) + 1 \end{aligned}$$

and no set of independent edges of  $H^*$  can consist of more than  $n-s$  edges. Provided the maximum degree is not increased, edges can be adjoined to this example to yield further examples.

When  $r = 1$  or  $r = 2$ , Conjecture 5.1 is equivalent to the following conjecture.

Conjecture 5.2 Let  $1 \leq r \leq n$ . Let  $G$  be a regular multigraph on  $2n+2$  vertices of degree  $2n+1-r$  in which all non-simple edges are on the same vertex. Let  $G$  not contain a subgraph on three vertices with  $2n+2-r$  edges. Then  $\chi'(G) = 2n+1-r$ .

In the case when  $G$  is simple,  $r = 3$  and the complement of  $G$  is the union of three 1-factors, Conjecture 5.2 has been verified by Rosa and



Wallis [R1]. This is also proved when  $G$  is simple and  $(2n+1-r) \geq \frac{6}{7}(2n+2)$  in Chapter 9.

Conjecture 5.1 implies the following conjecture.

Conjecture 5.3 Let  $1 \leq r \leq n$ . Let  $G$  be a critical graph with  $2n+1$  vertices and maximum degree  $2n+1-r$ . Then

$$|E(G)| = \binom{2n+1}{2} - rn + 1.$$

For  $2 < r \leq n$  and for graphs of maximum degree  $2n+1-r$  with  $2n+1$  vertices, Conjecture 5.3 is stronger than the conjecture of Vizing referred to in the introduction to Chapter 4. Whereas in Conjecture 5.3 a critical graph has

$$2n^2 - (r-1)n + 1$$

edges, according to the Vizing conjecture a critical graph has at least

$$2n^2 - (r-1)n - \frac{1}{2}(r-3)$$

edges. For  $1 \leq r \leq 2$ , the two conjectures coincide.

The restriction  $r \leq n$  in Conjectures 5.1, 5.2 and 5.3 would be best possible, as the example of two disjoint  $K_{n+1}$ 's, when  $n$  is even, shows that  $r$  cannot be increased to  $n+1$  in Conjecture 5.2.

We also have the following conjecture.

Conjecture 5.4. If a regular multigraph on  $2n$  vertices of degree  $2n-1$  has no submultigraph consisting of three vertices and  $2n+1$  edges, then it can be edge-coloured with  $2n$  colours.

Of course, Conjecture 5.4 is true for simple graphs.

## 6. The chromatic index of graphs of even order with many edges.

### 6.1 Introduction

Following the results on odd order graphs in Chapter 5 we now consider even order graphs. We show:

Theorem 6.1 For  $r = 1$  or  $2$  a graph  $G$  with  $2n+2$  vertices and maximum degree  $2n+1-r$  is of Class 2 if and only if

$$|E(G-v)| > \binom{2n+1}{2} - rn,$$

where  $v$  is a vertex of minimum degree.

We make a conjecture for  $1 < r < n$  of which this result is a special case. For  $r = 1$  this result is due to Plantholt [P2].

### 6.2 Even order graphs with large degree

Proof of Theorem 6.1. Since  $\binom{2n+1}{2} - rn = (2n+1-r)n$ , it is easy to see that the inequality is sufficient for  $G$  to be of Class 2. We shall now prove the necessity.

Assume that  $G$  is a simple graph with  $2n+2$  vertices, maximum degree  $\Delta(G) = 2n+1-r$ , such that  $|E(G-w)| < \binom{2n+1}{2} - rn$  ( $\forall w \in V(G)$ ), and  $r = 1$  or  $2$ .

We first create a graph  $H$  on  $2n+2$  vertices by adding in as many edges as possible to  $G$  in such a way that

$$(i) \quad |E(H-w)| \leq \binom{2n+1}{2} - rn \quad (\forall w \in V(H)),$$

$$(ii) \quad \Delta(H) = 2n+1-r, \text{ and}$$

(iii)  $H$  is a simple graph.

Let the vertices of  $H$  of degree less than  $2n+1-r$  be called  $v_1, \dots, v_p$ .

Let the remaining vertices of  $H$  be  $v_{p+1}, \dots, v_{2n+2}$ . Then either

$$|E(H \setminus v^*)| = \binom{2n+1}{2} - rn$$

for some vertex  $v^*$ , or

$$|E(H \setminus w)| < \binom{2n+1}{2} - rn \quad (\forall w \in V(H)),$$

$v_1, \dots, v_p$  are all joined to each other, and  $2n \geq p \geq 2$ .

(If  $p=0$  or  $1$  then for at least one vertex  $v^*$   $|E(H \setminus v^*)| \geq \binom{2n+1}{2} - rn$ ,

hence  $p \geq 2$ . To see that  $2n \geq p$ , notice that if  $p = 2n+1$  or  $2n+2$  then the degrees of  $v_1, \dots, v_p$  would be  $\geq 2n$ , a contradiction.)

Suppose first that, for some vertex  $v^*$ ,

$$|E(H \setminus v^*)| = \binom{2n+1}{2} - rn.$$

Then, since Conjecture 5.1 is true for  $r = 1$  and  $r = 2$ ,  $H \setminus v^*$  can be edge-coloured with  $2n + 1 - r$  colours. Since  $|E(H \setminus v^*)| = n(2n + 1 - r)$ , each colour class occurs on  $n$  edges. Therefore each colour is missing in  $H \setminus v^*$  from exactly one vertex. Therefore  $v^*$  and the edges of  $H$  on it can be adjoined with the edges receiving distinct colours. This yields an edge-colouring of  $H$  with  $2n + 1 - r$  colours and, therefore, an edge-colouring of  $G$  with  $2n + 1 - r$  colours.

Next suppose that

$$|E(H \setminus w)| < \binom{2n+1}{2} - rn \quad (\forall w \in V(H)),$$

$v_1, \dots, v_p$  are all joined to each other, and  $p \geq 2$ . We now add edges to  $H$  onto the vertices  $v_1, \dots, v_p$ , creating in each case multiple edges, but not loops, so as to form a regular multigraph  $H^*$  of degree  $2n + 1 - r$  on  $2n + 2$  vertices. To see that this can actually be done:

For  $1 \leq i \leq 2n + 2$ , let  $\delta_i = 2n + 1 - r - d_H(v_i)$ . We may assume that

$\delta_1 \geq \delta_2 \geq \dots \geq \delta_p$  (if not, then we may relabel the vertices  $v_1, \dots, v_p$ ).

We have to show that there is a loopless multigraph with degree

sequence  $(\delta_1, \delta_2, \dots, \delta_p)$ . By a result of Hakimi [H3], there is such a

loopless multigraph if and only if

$$(mi) \sum_{i=1}^p \delta_i \text{ is even, and}$$

$$(mii) \delta_1 \leq \sum_{i=2}^p \delta_i.$$

In our case, (mi) is satisfied since

$$\sum_{i=1}^p \delta_i = (2n+2)(2n+1-r) - \sum_{i=1}^{2n+2} d_H(v_i),$$

which is even since  $H$  is a graph. Also (mii) is satisfied, since

$$\begin{aligned} & \sum_{i=2}^p \delta_i - \delta_1 \\ &= \{(2n+1)(2n+1-r) - 2|E(H \setminus v)| - d_H(v_1)\} - \{(2n+1-r) - d_H(v_1)\} \\ &= 2n(2n+1-r) - 2|E(H \setminus v_1)| \\ &> 2n(2n+1-r) - 2\left\{\binom{2n+1}{2} - rn\right\}, \text{ by our assumption,} \\ &= 0. \end{aligned}$$

From  $H^*$  delete a vertex  $x$ ,  $x \in \{v_{p+1}, \dots, v_{2n+2}\}$ . Then  $x$  is joined to at least  $p-r$  of  $v_1, \dots, v_p$ , has no multiple edges on it and  $H^* \setminus x$  has  $r$  vertices of maximum degree  $2n+1-r$ . When  $r=2$ , if possible select  $x$  so that, in addition, in  $H^* \setminus x$ , at least one of the two vertices of maximum degree is in  $\{v_{p+1}, \dots, v_{2n+1}\}$ . If this is not possible, then, again if possible, in addition select  $x$  so that there exists a vertex  $y \in \{v_{p+1}, \dots, v_{2n+2}\}$  such that  $y$  is adjacent to one of the two vertices of maximum degree in  $H^* \setminus x$  and is non-adjacent to the other. The case when we cannot choose  $x$  to satisfy in addition either of these possibilities is considered at the end.

If  $r = 1$ , or if  $r = 2$  and one of the two additional requirements for the selection of  $x$  can be satisfied, then we now proceed to show that  $H^* \setminus x$  can be edge-coloured with  $2n + 1 - r$  colours. First we edge-colour the complete subgraph of  $H^* \setminus x$  on the vertices  $v_1, \dots, v_p$  with  $p$  colours if  $p \geq 3$  or with one colour if  $p = 2$ . We show below that  $|E(H^*)| - |E(H)| \leq \begin{cases} 2n + 1 - r - p & \text{if } p \geq 3 \\ 2n + 2 - r - p & \text{if } p = 2. \end{cases}$  Therefore, giving the edges of  $E(H^*) \setminus E(H)$  at most this number of further colours, we use at most  $2n + 1 - r$  colours in all on the maximal submultigraph of  $H^* \setminus x$  on  $v_1, \dots, v_p$ . To see that  $|E(H^*)| - |E(H)| \leq 2n + 1 - r - p$ , consider the graph  $\bar{H}$  (the complement of  $H$ ). Each vertex  $v$  with  $d_{\bar{H}}(v) > r$  is joined in  $\bar{H}$  solely to vertices of degree  $r$  in  $\bar{H}$ . Therefore the total number of edges in  $\bar{H}$  joining vertices of degree  $> r$  in  $\bar{H}$  (i.e.  $v_1, \dots, v_p$ ) to vertices of degree  $r$  in  $\bar{H}$  is at most  $(2n + 2 - p)r$ . Therefore

$$\begin{aligned} \sum_{i=1}^p \delta_i &= \sum_{i=1}^p (d_{\bar{H}}(v_i) - r) \\ &= \left\{ \sum_{i=1}^p d_{\bar{H}}(v_i) \right\} - rp \\ &\leq (2n + 2 - p)r - rp \\ &= 2(n + 1 - p)r. \end{aligned}$$

Therefore the number of edges of  $E(H^*) \setminus E(H)$  is

$$\begin{aligned} \frac{1}{2} \left( \sum_{i=1}^p \delta_i \right) &\leq (n + 1 - p)r \\ &\leq 2n + 2 - 2p \\ &\leq \begin{cases} 2n+1-r-p & \text{if } p \geq 3 \text{ and } r \leq 2 \\ 2n+2-r-p & \text{if } p = 2 \text{ and } r \leq 2. \end{cases} \end{aligned}$$

Next we colour  $H^* \setminus \{v_1, \dots, v_p, x\}$  with  $2n + 1 - r$  colours. This can be done since  $2n + 1 - r \geq 2n - 1$  and the number of vertices of  $H^* \setminus \{v_1, \dots, v_p, x\}$  is  $2n + 2 - p - 1 = 2n + 1 - p$ , which is at most  $2n - 1$  since  $p \geq 2$ .

Finally we colour the edges joining the maximal submultigraph of  $H^* \setminus x$  on  $v_1, \dots, v_p$  to the subgraph  $H^* \setminus \{v_1, \dots, v_p, x\}$ . This can be done by Vizing's original argument [V1] if the pivot vertex is always in  $H^* \setminus \{v_1, \dots, v_p, x\}$  and if the final edge to be coloured is as follows. If  $r = 1$  then the final edge must be incident with the vertex of maximum degree. If  $r = 2$ , let the two vertices of maximum degree be  $b$  and  $c$ . If at least one of these, say  $c$ , is in  $H^* \setminus \{v_1, \dots, v_p, x\}$ , then, if  $b$  and  $c$  are joined, colour the edge  $bc$  last. If  $b$  and  $c$  are not joined, then colour an edge on  $b$  next to last, and colour an edge on  $c$  last. If both  $b$  and  $c$  are in the set  $\{v_1, \dots, v_p\}$  but there is a vertex  $y \in \{v_{p+1}, \dots, v_{2n+2}\}$  joined to  $b$  but not to  $c$ , colour an edge on  $c$  next to last, and the edge  $yb$  last. Vizing's original argument will apply as, at any stage during the construction of the fans, there will always be a further colour available on the vertex at the other end from the pivot of the most recently adjoined edge; except at the final stage, such a further colour is then used to define the next stage of the fan. This yields the desired colouring of  $H^* \setminus x$  with  $2n + 1 - r$  colours. From this we obtain an edge-colouring of  $G$  with  $2n + 1 - r$  colours by the same argument as when (1) applied (with  $H^*$  and  $x$  instead of  $H$  and  $v$ ).

It remains to consider the case when  $x$  cannot be chosen to satisfy either of the additional conditions. Since it is not possible to select  $x$  so that, in addition to the other requirements, in  $H^* \setminus x$ , at least one of the two vertices of maximum degree is in  $\{v_{p+1}, \dots, v_{2n+2}\}$ , it follows that the subgraph of  $H^*$  induced by  $\{v_{p+1}, \dots, v_{2n+2}\}$  is complete. We already know that the sub(simple)graph of  $H^*$  induced by  $\{v_1, \dots, v_p\}$  is complete. Since  $\Delta(H^*) = 2n + 1 - r$ , in this case it follows that each vertex of  $\{v_{p+1}, \dots, v_{2n+2}\}$  is joined to all but two of  $\{v_1, \dots, v_p\}$ . Since it is not possible to select  $x$  so that, in addition to the other requirements, there is a vertex  $y \in \{v_{p+1}, \dots, v_{2n+2}\}$  such that  $y$  is adjacent to one of the two vertices of maximum degree in  $H^* \setminus x$  and non-

adjacent to the other, it now follows that  $p$  is even and that  $\{v_1, \dots, v_p\}$  and  $\{v_{p+1}, \dots, v_{2n+2}\}$  have the property that, after re-ordering if necessary, there are integers  $r_0 = 0, r_1, \dots, r_{p/2} = 2n + 2 - p$  such that, for  $1 \leq i \leq \frac{p}{2}$ ,  $v_{2i-1}$  and  $v_{2i}$  are both non-adjacent to  $v_{p+r_{i-1}+1}, \dots, v_{p+r_i}$  but are both adjacent to the rest of  $\{v_{p+1}, \dots, v_{2n+2}\}$ , and that  $r_i - r_{i-1} \geq 3$ .

Suppose  $p \geq 4$  and consider the graph  $H^*$ . We can show that this graph has a 1-factor. Consider  $H^* \setminus \{v_1, v_2, v_{p+r_1+1}, v_{p+r_1+2}\}$ . This graph contains two complete subgraphs on  $\{v_3, \dots, v_p\}$  and  $\{v_{p+1}, \dots, v_{p+r_1}, v_{p+r_1+3}, \dots, v_{2n+2}\}$  which will each have a 1-factor. If we take these two 1-factors together with the edges  $v_1 v_{p+r_1+1}$  and  $v_2 v_{p+r_1+2}$  then we have a 1-factor  $F$  of  $H^*$  containing the edge  $v_1 v_{p+r_1+1}$ .

Now consider  $H^* \setminus F$ . The same argument we applied to  $H^* \setminus x$  earlier, now applies (with trivial modifications) to  $(H^* \setminus F) \setminus v_{p+1}$ , since the sub(simple) graph of  $(H^* \setminus F) \setminus v_{p+1}$  induced by  $v_1, \dots, v_p$  can be coloured with  $p - 1$  colours, as  $p$  is even, and since the vertices  $v_1, v_2$  have maximum degree in  $(H^* \setminus F) \setminus v_{p+1}$ , and  $v_{p+1+r_1}$  is adjacent in it to  $v_2$  but non-adjacent in it to  $v_1$ . Consequently  $H^* \setminus F$  is Class 1, therefore so is  $H^*$ , and so it follows that  $G$  is Class 1.

Finally suppose that  $p = 2$ . Then  $H^*$  is  $K_{2n}$  plus two more vertices joined by  $2n-1$  multiple edges. Hence  $H^*$  is Class 1 and working back  $G$  is Class 1.

This proves Theorem 6.1.

### 6.3 Conjectures.

Conjecture 6.1. Let  $1 \leq r \leq n$ . Let  $G$  be a simple graph with  $2n+2$  vertices and maximum degree  $\Delta(G) = 2n+1-r$ . Then  $G$  is Class 2 if and only if for some  $s$  such that  $0 \leq s \leq \frac{r-1}{2}$  and for some set  $\{v_1, \dots, v_{2s+1}\} \subset V(G)$ ,  $|E(G) \setminus \{v_1, \dots, v_{2s+1}\}| > \binom{2n+1-2s}{2} - (r-2s)(n-s)$ .

When  $r = 1$  or  $2$  then  $s = 0$  and the last inequality becomes  $|E(G \setminus v)| > \binom{2n+1}{2} - rn$ . From Theorem 6.1, Conjecture 6.1 is true for  $r = 1$  and for  $r = 2$ . The case  $r = 1$  was proved earlier by Plantholt [P2].

In Chapter 5 we formulated a similar conjecture, Conjecture 5.1, for graphs of odd order. If Conjecture 5.1 is true then Conjecture 6.1 can be reformulated.

Conjecture 6.1\*. Under the conditions of Conjecture 6.1,  $G$  is Class 2 if and only if  $G \setminus v$  is Class 2 and  $\Delta(G \setminus v) = \Delta(G)$ .

Jakobsen [J4] and independently, Beineke and Wilson [B2] conjectured that all critical graphs have an odd number of vertices. This conjecture has recently been disproved by Gol'dberg [G3]. However if Conjecture 6.1\* were true, it would imply the following modified version of the Critical Graph Conjecture.

Conjecture 6.3. There are no critical graphs of order  $2n+2$  and maximum degree at least  $n+1$ .

A corollary to Theorem 6.1 is therefore:

Corollary 6.1. There are no critical graphs of order  $2n+2$  and maximum degree  $2n$  or  $2n-1$ .

In order to see that there must be some bound on the value of  $r$  in Conjecture 6.1, let  $G$  be the even order critical graph discovered by Gol'dberg. Then  $|V(G)| = 22$ ,  $\Delta(G) = 3$ ,  $|E(G)| = 31$  and  $r = 18$ . Then the inequality in Conjecture 6.1 would give

$$|E(G \setminus v)| > \binom{2n+1}{2} - rn = \binom{21}{2} - 18 \cdot 10 = 30.$$



But in Gol'dberg's graph,  $|E(G \setminus v)|$  is 29 for a vertex  $v$  of minimum degree and hence would be a counterexample for a large value of  $r$ .

7. Chromatic class of graphs with a given order and at most 4 vertices of maximum degree.

7.1. Introduction and statement of results.

In this chapter and the next we investigate the chromatic class of graphs when the number of vertices of maximum degree is fixed. If a graph  $G$  has just one or two vertices of maximum degree, then only a slight development of the proof of Vizing's theorem is needed to show that  $G$  is Class 1. However, if  $G$  has three vertices of maximum degree all joined to each other, then the proof of Vizing's theorem does not seem to lend itself to be adapted to prove some analagous result. The key step which inspired the next three chapters was the proof of the following proposition.

Proposition. Let  $G$  be a connected graph with three vertices of maximum degree. Then

$G$  is Class 2

if and only if

for some  $n$ ,  $G$  is obtained from  $K_{2n+1}$  by removing  $n-1$  independent edges.

In this chapter we develop this theme for graphs with up to four vertices of maximum degree.

We have the following result for graphs of even order.

Theorem 7.1 Let  $G$  be a graph of order  $2n$  with  $r$  vertices of maximum degree. If  $1 \leq r \leq 4$  then  $G$  is not critical.

We conjecture the following:

Conjecture 7.1. Theorem 7.1 is true for  $1 < r < n$ .

Our proof of Theorem 7.1 is rather long and complicated. We suspect that a similar proof would work to prove Conjecture 7.1 when  $r = 5$ , but would fail when  $r \geq 6$ . However, we were deterred by the amount of work

involved.

We also have a corresponding result for graphs of odd order.

Theorem 7.2. Let  $G$  be a graph of order  $2n+1$  with  $r$  vertices of maximum degree. If  $1 \leq r \leq 4$ , then the following are equivalent:

- (i)  $G$  is critical,
- (ii)  $|E(G)| = n\Delta + 1,$
- (iii)  $G$  is  $(r-2)$ -edge connected and Class 2.

Conjecture 7.2. Let  $G$  be a graph of order  $2n+1$  with  $r$  vertices of maximum degree. If  $1 \leq r \leq n+1$  then the following are equivalent:

- (i)  $G$  is critical,
- (ii)  $|E(G)| = n\Delta + 1,$
- (iii)  $G$  is  $(r-2)$ -edge-connected and Class 2, and  $|E(G)| \leq n\Delta + 1.$

In this case, our proof of Theorem 7.2 will not extend to give a proof of Conjecture 7.2 when  $r = 5$ . The reason is that in the case when  $r = 4$ , one can deduce from Lemma 7.2 that  $\Delta(G) \geq \frac{1}{2}|V(G)|$ , and the number  $\frac{1}{2}|V(G)|$  is compatible with the applications we have to make of Dirac's theorem (Lemma 7.5). However in the case when  $r = 5$ , one can only deduce from Lemma 7.2 that  $\Delta(G) \geq \frac{2}{5}|V(G)|$ . (In the case of Conjecture 7.1 when  $|V(G)|$  is even, it turns out that when  $r = 5$  the inequality  $\Delta(G) \geq \frac{1}{2}|V(G)|$  (actually  $\Delta(G) \geq \frac{1}{2}(|V(G)| - 1)$ ) can always be obtained from Lemma 7.3, as the graph  $G$  of even order with  $r = 5$  satisfies  $\delta(G) \leq \Delta(G) - 2$ , so we can put  $s \geq 2$  in that Lemma.)

The bound  $r \leq n+1$  comes from the fact that if  $|E(G)| = n\Delta + 1$  then the edge-connectivity is at least  $2n - r + 2$  (see Lemma 8.11), so if  $2n - r + 2 \geq r - 2$  then  $n + 2 \geq r$ . We have to exclude  $r = n + 2$  since the critical subgraph of the Petersen graph would give a counterexample.

A simple corollary to Theorem 7.2 is the following.

Theorem 7.3. Let  $G$  be an  $(r-2)$ -edge-connected graph of order  $2n+1$  with  $r$  vertices of maximum degree. If  $1 \leq r \leq 4$  then  $G$  is Class 2 if and only if

$$|E(G)| > n\Delta.$$

The following conjecture would follow from Conjecture 7.1 and 7.2.

Conjecture 7.3. Let  $1 \leq r \leq n+1$ . Let  $G$  be an  $(r-2)$ -edge-connected graph of order  $2n+1$  with  $r$  vertices of maximum degree  $\Delta$ . Then  $G$  is Class 2 if and only if  $|E(G)| > n\Delta$ .

A simple corollary to Theorems 7.1 and 7.2 is :

Theorem 7.4. Let  $G$  be an  $(r-2)$ -edge-connected graph of order  $2n$  with  $r$  vertices of maximum degree. If  $1 \leq r \leq 4$ , then  $G$  is Class 1.

The corresponding conjecture would be :

Conjecture 7.4. Let  $1 \leq r \leq n$ . Let  $G$  be an  $(r-2)$ -edge-connected graph of order  $2n$  with  $r$  vertices of maximum degree  $\Delta$ . Then  $G$  is Class 1.

Class 2 graphs with three vertices of maximum degree are fully described in the proposition. A similar description is possible for Class 2 graphs with four vertices of maximum degree.

Theorem 7.5. Let  $G$  be a connected graph with four vertices of maximum degree. Then

$G$  is Class 2

if and only if

$G$  is one of the following graphs, for some  $n$ ,

(i)  $G \cong (2n-2)^{2n-3} (2n-1)^4,$

(ii)  $G \cong (2n-2) (2n-1)^{2n-4} (2n)^4,$

(iii) for some  $m < n$ ,  $G$  has a bridge  $e$ ; one component  $C_1$  of  $G \setminus e$  has maximum degree at most  $2m-1$  and, in  $G$ ,  $e$  is incident with a vertex of degree in  $C_1$  at most  $2m-2$ ; and the other component  $C_2$  satisfies

$$C_2 \cong (2m-2) (2m-1)^{2m-4} (2m)^4$$

or

$$C_2 \cong (2m-1)^{2m-2} (2m)^3.$$

## 7.2. Some preliminary lemmas.

Before embarking on the proofs of these results, we need to establish a number of lemmas.

Lemma 7.1. Let  $G$  be a graph of order  $2n+1$  with  $r$  vertices of maximum degree  $\Delta$  and with  $|E(G)| = n\Delta + 1$ . Then  $\Delta \geq 2n+3 - r$ .

Proof.  $\Delta(G)n+1$

$$\begin{aligned} &= |E(G)| \\ &\leq \frac{1}{2} \{r\Delta(G) + (2n+1-r)(\Delta(G)-1)\} \\ &= n\Delta(G) + \frac{1}{2} (\Delta(G) - 2n-1+r), \end{aligned}$$

so

$$2n-r+3 \leq \Delta(G).$$

Corollary 7.1. Under the conditions of Lemma 7.1,

$$\max(r, \Delta(G)) \geq n+2.$$

Lemma 7.2. Let  $G$  be a critical graph with  $r$  vertices of maximum degree  $\Delta(G)$ . Then

$$\Delta(G) \geq \frac{2|V(G)|}{r}.$$

Proof. By Lemma 2.2, each vertex is joined to at least two vertices of degree  $\Delta(G)$ . Each vertex of maximum degree is joined to  $\Delta(G)$  other vertices. Therefore  $2|V(G)| \leq \Delta(G)r$ , and the result follows.

Lemma 7.3. Let  $r \geq s \geq 1$ . Let  $G$  be a critical graph. If  $G$  has  $r$  vertices of maximum degree  $\Delta(G)$  and at least one vertex of degree  $\Delta(G)-s$ , then

$$\Delta(G) \geq \frac{2|V(G)| - s(s-1)}{r - s + 1}.$$

Proof. The vertex of degree  $\Delta(G)-s$  is adjacent to  $\Delta(G)-s$  vertices, each of which is, by Lemma 2.1, adjacent to at least  $s+1$  vertices of maximum degree. The remaining  $|V(G)| - \Delta(G) + s$  vertices are, by Lemma 2.2, adjacent to at least two vertices of maximum degree. Counting the edges incident with vertices of maximum degree, we have

$$(s+1)(\Delta(G)-s) + 2(|V(G)| - \Delta(G) + s) \leq r \Delta(G)$$

from which the lemma follows.

The next lemma is an extension due to Berge [B3] of a well-known theorem of Chvatal [C10].

Lemma 7.4. Let  $G$  be a simple graph of order  $n$  with degrees  $d_1 \leq d_2 \leq \dots \leq d_n$ . Let  $q$  be an integer,  $0 \leq q \leq n-3$ . If, for every  $k$  with  $q \leq k \leq \frac{1}{2}(n+q)$ , the following condition holds:

$$d_{k-q} \leq k \implies d_{n-k} \geq n-k+q$$

then, for each set  $F$  of independent edges with  $|F| = q$ , there exists a Hamiltonian circuit containing  $F$ .

A special case of Lemma 7.4 which we shall make much use of is the following result of Dirac [D2].

Lemma 7.5. Let  $G$  be a simple graph. If

$$\delta(G) \geq \frac{1}{2}|V(G)|$$

then G has a Hamiltonian circuit.

The next lemma is a nice result of Jackson [ J1 ].

Lemma 7.6. Every 2-connected, k-regular graph on at most  $3k$  vertices is Hamiltonian.

When  $r = 1$  the next result is due to Plantholt [P1] and when  $r = 2$  it is proved in Theorem 5.1.

Lemma 7.7. Let  $1 \leq r \leq 2$ . Let G be a graph of order  $2n+1$  with  $\Delta(G) = 2n+1-r$  and  $|E(G)| \leq (2n+1-r)n$ . Then G is Class 1.

The next result is due to Bollobás and Eldridge [B5].

Lemma 7.8. Let G be a graph with order n, maximum degree  $\Delta$ , minimum degree  $\delta$ . Then G contains at least  $m_0(n, \delta, \Delta)$  independent edges, where,

$$m_0(n, \delta, \Delta) = \begin{cases} \min \{ \lfloor n/2 \rfloor, \delta \} & \text{if } \delta \leq \Delta-2 \text{ and } n < \Delta + \delta, \\ \left\lfloor \frac{n\delta}{\delta+\Delta} \right\rfloor & \text{if } \delta \leq \Delta-2 \text{ and } n \geq \Delta + \delta, \\ \left\lfloor \frac{n\Delta}{2(\Delta+1)} \right\rfloor & \text{if } \delta = \Delta \text{ even, or } \delta = \Delta-1 \text{ odd,} \\ \left\lfloor \frac{n\delta+1}{2(\delta+1)} \right\rfloor & \text{if } \delta = \Delta-1 \text{ even,} \\ \frac{1}{2} n & \text{if } \delta = \Delta \text{ odd and } n = \delta + 1. \end{cases}$$

Finally when  $\delta = \Delta$  odd and  $n > \delta+1$  there are integers  $u, k, r$  such that

$$n = u(\delta+1)^2 + (2k+1)(\delta+2) + r \text{ and } 0 \leq 2k < \delta, 1 \leq r \leq 2\delta + 3$$

and  $m_0(n, \delta, \Delta) = \frac{1}{2} \{n - u(\delta-1)\} - k$ .

### 7.3. Proof of the proposition.

Here we prove the proposition given in Section 7.1.

Proof.

Sufficiency. If  $G$  has three vertices of degree  $|V(G)| - 1$  and the rest have degree  $|V(G)| - 2$ , then it is easy to check that  $|V(G)|$  is odd and

$$|E(G)| > \Delta(G) \left\lfloor \frac{|V(G)|}{2} \right\rfloor.$$

Each colour class cannot contain more than  $\left\lfloor \frac{|V(G)|}{2} \right\rfloor$  edges. Thus more than  $\Delta(G)$  colours are needed to colour the edges of  $G$ , so  $G$  is Class 2.

Necessity. Suppose  $G$  has three vertices  $a, b, c$  of maximum degree and is Class 2; but suppose also that  $G$  is not a graph with three vertices of degree  $|V(G)| - 1$  and  $|V(G)| - 3$  vertices of degree  $|V(G)| - 2$ . We may assume that  $|V(G)| \geq 4$ , as the necessity is clearly true if  $|V(G)| = 3$ .

Let  $G^*$  be a critical subgraph of  $G$  with  $\Delta(G) = \Delta(G^*)$ . By Lemma 2.5,  $G^*$  has at least three vertices of maximum degree, and, since  $\Delta(G) = \Delta(G^*)$ ,  $G^*$  has the same three vertices  $a, b, c$  of maximum degree.

By Lemma 2.6,  $G^*$  contains  $|V(G^*)| - 3$  vertices of degree  $\Delta(G^*) - 1$ . Therefore  $|V(G^*)|$  is odd, say  $|V(G^*)| = 2p + 1$ . By Lemma 2.2,  $a, b, c$  are all joined to each other in  $G^*$ . By Lemma 7.2,

$$\Delta(G^*) \geq \frac{2}{3} |V(G^*)|. \quad (1)$$

If there is a vertex  $u$  in  $G$  which is not in  $G^*$ , then  $u$  would not be joined to any of  $a, b, c$  (for otherwise  $G^*$  would have to have fewer than three vertices of maximum degree), but neither would  $u$  be joined to any vertex of  $V(G) \setminus \{a, b, c\}$  (for otherwise,  $G^*$  would have to have some vertices of degree  $\leq \Delta(G^*) - 2$ ). Thus there cannot be any such vertex  $u$ . Therefore  $G = G^*$ .

By the results of Chapter 3, the necessity is true for  $p \leq 4$ . Therefore suppose that  $p \geq 5$ .



Since  $G = G^*$ ,  $|V(G)| = 2p + 1$  and  $\Delta(G) \leq 2p - 1 = |V(G)| - 2$ . Therefore there is a vertex  $d$  not joined to  $a$ . Consider the graph  $G \setminus \{a, b\}$ . For  $p \geq 5$ , using (1),

$$\begin{aligned} \delta(G \setminus \{a, b\}) &\geq \Delta(G) - 3 \\ &> \left\lfloor \frac{4p + 2}{3} \right\rfloor - 3 \\ &> \frac{1}{2}(2p - 1) \\ &= \frac{1}{2} |V(G \setminus \{a, b\})|. \end{aligned}$$

By Lemma 7.5,  $G \setminus \{a, b\}$  has a Hamiltonian circuit. Therefore  $G$  has a near 1-factor  $F$  which contains the edge  $ab$  but does not include an edge incident with  $d$ . Therefore  $G \setminus F$  has 4 vertices,  $a, b, c, d$  of maximum degree, joined as illustrated in Figure 7.1.

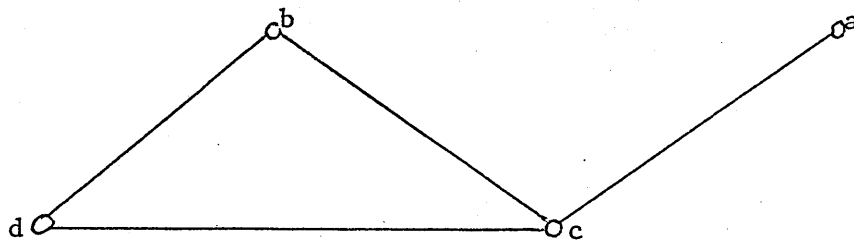


Figure 7.1.

Since  $d_{G \setminus F}^*(a) = 1$  and  $\Delta(G \setminus (F \cup ac)) = \Delta(G \setminus F)$ , by Lemma 2.4,  $G \setminus F$  is the same Class as  $G \setminus (F \cup ac)$ , which, by Lemma 2.5, is Class 1. Therefore  $G$  is Class 1.

Since  $\sum_{v \in V(G)} d(v) = 2|E(G)|$ , it is not possible for a graph of even order to have 3 vertices of degree  $\Delta$ , and the remaining vertices to have degree  $\Delta - 1$ . Therefore  $|V(G)|$  is odd.

This proves the proposition.

#### 7.4 Proof of Theorem 7.1.

Case 1.  $1 \leq r \leq 2$ . By Lemma 2.5, if  $G$  has 1 or 2 vertices of maximum degree, then  $G$  is Class 1. This proves the lemma in this case.

Case 2.  $r = 3$ . It follows immediately from the proposition that if  $|V(G)|$  is even, then  $G$  is Class 1. This proves the lemma in this case.

Case 3.  $r = 4$ . It follows from Beineke and Fiorini [Bf] that this lemma is true when  $n \leq 5$ . So we shall assume that  $n \geq 6$ .

Suppose that  $G$  is a critical graph. By Lemma 2.6,  $\delta(G) \geq \Delta(G) - 2$ . Therefore, for some integer  $x$ ,

$$G \cong (\Delta - 2)^x (\Delta - 1)^{|V(G)| - x - 4} \Delta^4.$$

By Lemma 7.3,

$$\Delta(G) \geq \frac{2(|V(G)| - 1)}{3} \quad \text{if } x \neq 0,$$

and by Lemma 7.2,

$$\Delta(G) \geq \frac{1}{2}|V(G)|.$$

Let the four vertices of maximum degree be  $a, b, c, d$ . We may assume that  $d^*(a) \leq d^*(v)$  for  $v \in \{a, b, c, d\}$ . By Lemma 2.2,  $d^*(a) \geq 2$ . We may assume that  $ab \in E(G)$ . We consider various cases.

Case 3i.  $\max_{v \in \{a, b, c, d\}} d^*(v) = 2$ .

Case 3i a.  $x \geq 1$ . Let  $d(z) = \Delta(G) - 2$ . By Lemma 2.2,  $d^*(z) \geq 2$ , so  $z$  is adjacent to at least two of  $\{a, b, c, d\}$ ; without loss of generality, assume that  $zc \in E(G)$ . Then, by Lemma 2.1,  $G$  is not critical unless  $d^*(c) \geq 3$ , a contradiction. Therefore Case 3ia does not arise.

Case 3i b.  $x = 0$ .

Case 3i b1.  $\Delta(G) \geq n + 2$ . Consider the graph  $G \setminus \{a, b\}$ . We have  $\delta(G \setminus \{a, b\}) \geq \Delta(G) - 3$  and  $|V(G \setminus \{a, b\})| = 2n - 2$ . Therefore by Lemma 7.5,  $G \setminus \{a, b\}$  contains a Hamiltonian cycle and hence  $G$  has a 1-factor  $F$  including the edge  $ab$ . The graph  $G \setminus F$  has 4 vertices  $a, b, c, d$  of maximum degree and  $d_{G \setminus F}^*(a) = 1$ . We may assume that  $a$  is joined to  $c$ . Then, by Lemma 2.4,  $G \setminus F$  and  $G \setminus (F \cup ac)$  have the same Class. But  $G \setminus (F \cup ac)$  has just two vertices of maximum degree and so, by Lemma 2.5, is Class 1. Working back it follows that  $G$  is Class 1.

Case 3i b2.  $\Delta(G) = n + 1$ . In this case,  $G \cong n^{2n-4}(n+1)^4$ . Since  $\delta(G) = n = \frac{1}{2} |V(G)|$ , it follows by Lemma 7.5 that  $G$  has a Hamiltonian cycle  $H$ . If  $H$  includes at least one edge between two vertices of degree  $n+1$  ( $a, b$  say), then we can take alternate edges of  $H$  including  $ab$  to be our 1-factor  $F$ , and argue as in Case 3i b1. Otherwise  $H$  does not include any edge between two vertices of degree  $n+1$ . Then, on the cycle  $H$ , each of  $\{a, b, c, d\}$  must have two adjacent vertices, neither of which is in  $\{a, b, c, d\}$ . Since the number of vertices is odd, there must be at least five vertices of less than maximum degree joined by edges of  $H$  to vertices of  $\{a, b, c, d\}$ . Since there are at most four vertices joined in  $G$  to more than two vertices of maximum degree, it is possible to form a 1-factor  $F$  by picking alternate edges of  $H$  so that  $F$  has at least one edge which joins a vertex  $v$  with  $d^*(v) = 2$  to a vertex of maximum degree. Hence in  $G \setminus F$ ,  $v$  is joined to only one vertex of maximum degree, so  $(G \setminus F) \setminus \{v\}$  has 3 vertices of maximum degree not all joined to each other and so, by the proposition,  $(G \setminus F) \setminus \{v\}$  is Class 1 and hence  $G$  is Class 1.

Case 3ib3.  $\Delta(G) = n$ . In this case,  $G \cong (n-1)^{2n-4} n^4$ . Therefore there are  $4n-8$  edges joining  $\{a,b,c,d\}$  to the rest of the graph. Since  $G$  is critical, every vertex is joined to at least two vertices of maximum degree; in this case, every vertex is joined to exactly two of maximum degree. Therefore  $G \setminus \{a,b,c,d\}$  is a regular graph of degree  $n-3$ . We can see that either this graph is 2-connected or it is two copies of  $K_{n-2}$ . If  $G \setminus \{a,b,c,d\}$  is two copies of  $K_{n-2}$  and  $n$  is even then  $G \setminus \{a,b,c,d\}$  has a 1-factor, and so  $G$  contains a 1-factor  $F$  including  $ab$ . The argument then proceeds as in Case 3ib1. If  $n$  is odd then it is possible to find two adjacent vertices of maximum degree, say  $a,b$ , such that  $G \setminus \{a,b\}$  has a 1-factor. Then let  $F$  consist of  $ab$  and this 1-factor, and argue as in Case 3ib1. Otherwise  $G \setminus \{a,b,c,d\}$  is 2-connected and so, by Lemma 7.6, if  $3(n-3) \geq 2n-4$ , then there exists a Hamiltonian cycle in  $G \setminus \{a,b,c,d\}$  and hence a 1-factor  $F$  in  $G$  containing  $ab$ , and we proceed as in Case 3ib1. But this inequality is satisfied since  $n \geq 6$ .

Case 3ib4.  $\Delta(G) < n$ . By Lemma 7.2, this case does not arise.

Case 3ii.  $d^*(a) = 2$  and  $\max_{v \in \{a,b,c,d\}} d^*(v) = 3$ .

Case 3ii a.  $x \geq 1$ .

Case 3ii a1.  $\Delta \geq n+2$ . Consider the graph  $G \setminus \{a,b\}$ . By Lemma 2.1, a vertex  $v$  of degree  $\Delta-2$  in  $G$  is only joined in  $G$  to vertices  $w$  such that  $d_G^*(w) \geq \Delta - (\Delta-2) + 1 = 3$ . Therefore no vertex joined to  $a$  has degree  $\Delta-2$ . Therefore  $\delta(G \setminus \{a,b\}) \geq \Delta-3 \geq (n+2)-3 = n-1 = \frac{1}{2}|V(G \setminus \{a,b\})|$ , so by Lemma 7.5,  $G$  contains a 1-factor  $F$  including  $ab$ .

Then  $G \setminus F$  has the same four vertices of maximum degree and  $d_{G \setminus F}^*(a) = 1$ ; we may assume that  $a$  is joined to  $c$ . Then, by Lemma 2.4,  $G \setminus F$  and  $G \setminus (FUac)$  have the same Class. But  $G \setminus (FUac)$  has just two vertices of maximum degree, so, by Lemma 2.5, is Class 1. Working back, it follows that  $G$  is Class 1.

Case 3ii a2.  $\Delta \leq n+1$ . By Lemma 7.3,  $n + 1 \geq \Delta \geq \frac{2(2n-1)}{3}$ . Therefore  $n \leq 5$ . But this contradicts our assumption that  $n \geq 6$ . Therefore this case does not arise.

Case 3ii b.  $x = 0$ .

Case 3ii b1.  $\Delta \geq n+2$ . In this case, since  $x = 0$ ,  $\delta(G \setminus \{a, b\}) \geq \Delta - 3$ . The argument now proceeds as in Case 3ii a1.

Case 3ii b2.  $\Delta(G) = n+1$ . There are  $4n-6$  edges from  $\{a, b, c, d\}$  to  $G \setminus \{a, b, c, d\}$ . It follows that the degree sequence of  $G \setminus \{a, b, c, d\}$  is either  $(n-4, n-2, n-2, \dots, n-2)$  or  $(n-3, n-3, n-2, \dots, n-2)$ . Since  $|V(G \setminus \{a, b, c, d\})| = 2n-4$ , it follows from Lemma 7.4 that  $G \setminus \{a, b, c, d\}$  has a Hamiltonian cycle. Therefore  $G$  has a 1-factor  $F$  which contains the edges  $ab$  and  $cd$ . Then, in  $G \setminus F$ ,  $d^*(a) = 1$  and hence  $(G \setminus F) \setminus a$  and  $G \setminus F$  have the same Class, by Lemma 2.4. But  $(G \setminus F) \setminus a$  has only 2 vertices of maximum degree and so, by Lemma 2.5, is Class 1. Hence  $G$  is Class 1.

Case 3ii b3.  $\Delta(G) \leq n$ . Since each vertex is joined to at least two of  $\{a, b, c, d\}$ , by counting the edges between  $\{a, b, c, d\}$  and  $V(G) \setminus \{a, b, c, d\}$ , it follows that

$$2(2n-4) \leq 2(\Delta-2) + 2(\Delta-3),$$

and so  $n + 1 \leq \Delta$ . Therefore this case does not arise.

Case 3iii a2.  $\Delta \leq n+2$ . By Lemma 7.3,

$$\frac{4n-2}{3} \leq \Delta \leq n+2.$$

It follows that  $n \leq 8$ . If  $n = 8$  then  $\Delta = 10$ , if  $n = 7$  then  $\Delta = 9$ , and if  $n = 6$  then  $\Delta = 8$ . In these cases we show that  $G$  has a 1-factor including the edge  $ad$ ; it then follows as in the previous case that  $G$  is Class 1.

We use Lemma 7.4 to show that  $G \setminus a$  has a Hamiltonian cycle. We then take  $ad$  and alternate edges of the cycle avoiding  $d$  to be the 1-factor. In  $G$  there are  $n-1$  edges joining  $a$  to vertices of  $V(G) \setminus \{a, b, c, d\}$ . Since  $\delta(G) = n$  there are at most  $n-1$  vertices of degree  $n-1$  in  $G \setminus a$ . Let  $d_1 \leq d_2 \leq \dots \leq d_{2n-1}$  be the degree sequence of  $G \setminus a$ . Therefore if  $d_{n-1} \leq n-1$  then  $d_n \geq n$ , and so, by Lemma 7.4,  $G \setminus a$  has a Hamiltonian cycle, as required.

Case 3iii b.  $x = 0$ .

Case 3iii b1.  $\Delta(G) \geq n+3$ . Consider the graph  $G \setminus \{a, b, c, d\}$ . We have  $\delta(G \setminus \{a, b, c, d\}) \geq \Delta(G) - 5$  and  $|V(G \setminus \{a, b, c, d\})| = 2n-4$ . If  $\Delta(G) - 5 \geq n-2$  then  $\delta(G \setminus \{a, b, c, d\}) \geq \frac{1}{2} |V(G \setminus \{a, b, c, d\})|$ , so by Lemma 7.5,  $G \setminus \{a, b, c, d\}$  contains a Hamiltonian cycle. Therefore  $G$  contains two 1-factors,  $F_1$  and  $F_2$ , such that  $F_1$  includes the edges  $ac$  and  $bd$ , and  $F_2$  includes the edges  $ad$  and  $bc$ . The graph  $G \setminus (F_1 \cup F_2)$  has four vertices,  $a, b, c, d$ , of maximum degree and  $d^*(v) = 1$  for each of them. By Lemma 2.4,  $G \setminus (F_1 \cup F_2)$  and  $(G \setminus a) \setminus (F_1 \cup F_2)$  have the same Class; but the latter graph has only two vertices of maximum degree, so, by Lemma 2.5, is Class 1. Therefore  $G$  is Class 1.

Case 3iii b2.  $\Delta(G) = n+2$ . In this case, we again consider the graph  $G \setminus \{a, b, c, d\}$ . The number of edges joining  $\{a, b, c, d\}$  to the rest of  $G$  is

4n-4. If  $d_1 \leq d_2 \leq \dots \leq d_{2n-4}$  is the degree sequence of  $G \setminus \{a, b, c, d\}$ , then we have  $d_5 = n-1$  and, since  $n \geq 6$ , it then follows from Lemma 7.4 that  $G \setminus \{a, b, c, d\}$  has a Hamiltonian circuit. Then the argument of Case 3iii b1 applies to show that  $G$  is Class 1.

Case 3iii b3.  $\Delta(G) = n+1$ . The number of edges joining  $\{a, b, c, d\}$  to  $G \setminus \{a, b, c, d\}$  is  $4n-8$ . Since  $|V(G \setminus \{a, b, c, d\})| = 2n-4$ , it follows by Lemma 2.2 that  $d^*(v) = 2$  for  $v \in V(G \setminus \{a, b, c, d\})$ , and so  $G \setminus \{a, b, c, d\}$  is regular of degree  $n-2$ . By Lemma 7.5,  $G \setminus \{a, b, c, d\}$  has a Hamiltonian cycle. Therefore  $G$  contains two edge-disjoint 1-factors,  $F_1$  and  $F_2$ , where  $F_1$  contains  $ab$  and  $cd$ , and  $F_2$  contains  $ad$  and  $bc$ . Then  $G \setminus (F_1 \cup F_2)$  has four vertices,  $a, b, c, d$ , of maximum degree, but each is adjacent to only one of the others. By Lemma 2.4,  $(G \setminus a) \setminus (F_1 \cup F_2)$  and  $G \setminus (F_1 \cup F_2)$  have the same Class. But  $(G \setminus a) \setminus (F_1 \cup F_2)$  has only two vertices of maximum degree, so by Lemma 2.5, is Class 1. Therefore  $G$  is Class 1.

This completes the proof of Theorem 7.1.

#### 7.5. Proof of Theorem 7.2.

Case 1.  $1 \leq r \leq 2$ . By Lemma 2.5, if  $G$  has 1 or 2 vertices of maximum degree, then  $G$  is Class 1. In this case the condition that if  $G$  is  $(r-2)$ -edge-connected is vacuous. Theorem 7.2 follows if we show that  $|E(G)| < n \Delta(G) + 1$ . But this follows from Lemma 7.1, since  $\Delta(G) \leq 2n$ .

Case 2.  $r = 3$ . We first prove that (iii)  $\Rightarrow$  (ii). If  $G$  is Class 2, then, by the proposition,  $G$  has three vertices of degree  $|V(G)| - 1$  and the remainder have degree  $|V(G)| - 2$ . Therefore

$$\begin{aligned} |E(G)| &= \frac{1}{2}(3 \cdot 2n + (2n-2)(2n-1)) \\ &= 2n^2 + 1 \\ &= n\Delta(G) + 1. \end{aligned}$$

This proves that (iii)  $\Rightarrow$  (ii).

To prove that (ii)  $\Rightarrow$  (iii), suppose that (ii) is true. Then by Lemma 2.7,  $G$  is Class 2. By Lemma 7.1,  $\Delta(G) = 2n$ ; consequently  $G$  is connected for all  $n \geq 1$ . This proves that (ii)  $\Rightarrow$  (iii).

To see that (i)  $\Rightarrow$  (iii) it is clear that if  $G$  is critical, then  $G$  is Class 2 and connected. By Lemma 2.7,  $|E(G)| < n\Delta + 1$ .

To prove that (iii)  $\Rightarrow$  (i), suppose that  $G$  satisfies (iii). Let  $G^*$  be a critical subgraph of  $G$  with the same maximum degree. Then, by the proposition,  $G$  has  $\Delta(G) = |V(G)| - 1$  and  $G^*$  has  $\Delta(G^*) = |V(G^*)| - 1$ . Therefore  $V(G) = V(G^*)$ . Since (iii)  $\Rightarrow$  (ii) we have  $|E(G^*)| = n\Delta(G^*) + 1 = n\Delta(G) + 1 = |E(G)|$ . Therefore  $G = G^*$ , so (i) is true.

This proves Theorem 7.2 in this case.

Case 3.  $r = 4$ . It follows from the results of Chapter 3 that this is true for  $n < 4$ . We shall therefore suppose that  $n \geq 5$ .

Suppose that  $G$  is a critical graph and that  $\left\lfloor \frac{|V(G)|}{2} \right\rfloor = n$ . We shall show that (ii) is satisfied. Suppose that (ii) is not satisfied. Then

$$|E(G)| < n\Delta(G).$$

If  $|V(G)| = 2n+1$  and  $\Delta(G) = 2n$  or  $2n-1$ , then the assumption that

$|E(G)| < n\Delta(G)$  means that the hypothesis of Lemma 7.7 is satisfied.

Therefore  $G$  is Class 1. From now on we shall suppose that  $\Delta(G) \leq 2n-2$ .

By Lemma 2.6,  $\delta(G) \geq \Delta(G) - 2$ . Therefore, for some integer  $x$ ,

$$G \cong (\Delta-2)^x (\Delta-1)^{|V(G)| - x - 4} \Delta^4.$$



By Lemma 7.3,

$$\Delta(G) \geq \frac{2(|V(G)| - 1)}{3} \quad \text{if } x \neq 0,$$

and by Lemma 7.2,

$$\Delta(G) \geq \frac{1}{2} |V(G)|.$$

Let the four vertices of maximum degree be  $a, b, c, d$ . We may assume that  $d^*(a) \leq d^*(v)$  for  $v \in \{a, b, c, d\}$ . By Lemma 2.2,  $d^*(a) \geq 2$ . We may assume that  $ab \in E(G)$ . We consider various cases.

Case 3i.  $\max_{v \in \{a, b, c, d\}} d^*(v) = 2$ .

Case 3i. a.  $x \geq 1$ . Let  $d(z) = \Delta(G) - 2$ . By Lemma 2.2,  $d^*(z) \geq 2$ , so  $z$  is adjacent to at least two of  $\{a, b, c, d\}$ ; without loss of generality, assume that  $zc \in E(G)$ . Then, by Lemma 2.1,  $G$  is not critical unless  $d^*(c) \geq \Delta(G) - d(z) + 1 = 3$ , a contradiction. Therefore Case 3ia does not arise.

Case 3i b.  $x = 0$ .

Case 3i b1.  $\Delta \geq n+3$ . Let  $v$  be a vertex not joined to  $a$  with  $d(v) = \Delta(G) - 1$ . If we can find a near 1-factor  $F$  of  $G$  which contains the edge  $ab$  but does not include any edge incident with  $v$ , then  $G \setminus F$  will have 5 vertices  $a, b, c, d$  and  $v$  of maximum degree, and  $d^*_{G \setminus F}(a) = 1$ . Then, by Lemma 2.4,  $G \setminus F$  and  $(G \setminus F) \setminus \{a\}$  have the same Class. Then we may assume that  $a$  is joined to  $c$  in  $G$ . Then  $(G \setminus F) \setminus \{a\}$  has 3 vertices,  $b, d, v$ , of maximum degree  $\Delta(G) - 1$ ,  $(\Delta(G) - 2)$  vertices of minimum degree  $\Delta(G) - 3$  and  $2n - 1 - \Delta(G)$  vertices of degree  $\Delta - 2$ . By the proposition,  $(G \setminus F) \setminus \{a\}$  could only be Class 2 if there were  $\Delta(G) - 3$  vertices of degree  $\Delta(G) - 2$  in one component of  $(G \setminus F) \setminus \{a\}$ , so that  $2n - 1 - \Delta(G) \geq \Delta(G) - 3$ ; i.e.  $\Delta(G) \leq n + 1$ .

Therefore provided  $\Delta(G) \geq n+2$ , then  $(G \setminus F) \setminus \{a\}$  is Class 1 and so  $G$  is Class 1.

Now consider  $G \setminus \{a, b\}$ . We have  $\delta(G \setminus \{a, b\}) \geq \Delta(G) - 3$  and  $|V(G \setminus \{a, b\})| = 2n-1$ . By Lemma 7.5, if  $\Delta(G) - 3 \geq \frac{1}{2}(2n-1)$  then there is a Hamiltonian cycle in  $G \setminus \{a, b\}$  and hence a suitable  $F$ . The inequality is true provided  $\Delta(G) \geq n + 2\frac{1}{2}$ .

Case 3i b2.  $\Delta(G) = n + 2$ . Then  $G \cong (n+1)^{2n-3} (n+2)^4$ . Let  $v$  be a vertex which is joined to one of  $\{a, b, c, d\}$ , say  $b$ , and not joined to another, say  $a$ . Then  $G \setminus \{v\}$  has  $2n$  vertices. Their degrees are  $n$ ,  $n+1$  and  $n+2$ , and at most  $n-1$  of them have degree  $n$ . Therefore if  $d_1 \leq d_2 \leq \dots \leq d_{2n}$  is the degree sequence of  $G \setminus \{v\}$ , then  $d_i \geq n$  ( $1 \leq i \leq n-1$ ) and  $d_i \geq n+1$  ( $n \leq i \leq 2n$ ). Using Lemma 7.4, it follows that  $G \setminus \{v\}$ , has a 1-factor  $F$  which includes  $ab$ . Therefore, as described in Case 3i b1,  $G$  is Class 1.

Case 3i b3.  $\Delta(G) = n+1$ .  $G$  has at most two vertices  $v$  such that  $d^*(v) \geq 3$ . Hence  $G \setminus \{a, b, c, d\}$  has either

$$A = (n-3, n-3, n-2, \dots, n-2)$$

or 
$$B = (n-4, n-2, n-2, \dots, n-2)$$

as its degree sequence.

If the degree sequence is  $A$ , we can use Lemma 7.8 to show that there is a near 1-factor in  $G \setminus \{a, b, c, d\}$ . Therefore  $G$  contains a near 1-factor which contains the edge  $ab$ , and it follows as in Case 3i b1 that  $G$  is Class 1.

If the degree sequence is  $B$ , we consider the graph  $G \setminus \{a, b, c, d, v_1\}$ , where  $v_1$  is the vertex of degree  $n-4$ . It follows from Lemma 7.8 that this graph has a 1-factor  $F$ . Since  $n \geq 5$  there is a vertex  $r$  in  $V(G \setminus \{a, b, c, d, v_1\})$  which is joined to  $v_1$ . Let  $rs \in F'$ . We may suppose

that  $sa \notin E(G)$ . Let  $F^* = (F' \setminus \{rs\}) \cup \{rv_1\} \cup \{ab\} \cup \{cd\}$ . The graph  $G \setminus F^*$  has five vertices of maximum degree, but  $d_{G \setminus F^*}^*(a) = 1$ , and so, by Lemma 2.4,  $G \setminus F^*$  and  $(G \setminus F^*) \setminus \{ac\}$  have the same Class. [Here we assume that  $ac \in E(G)$ ]. But similarly,  $(G \setminus F^*) \setminus \{ac\}$  and  $(G \setminus F^*) \setminus (\{ac\}, \{cs\})$  have the same Class; however this latter graph has only two vertices of maximum degree, and so, by Lemma 2.5, is Class 1. Working back, it follows that  $G$  is Class 1.

Case 3ii.  $d^*(a) = 2$  and  $\max_{v \in \{a,b,c,d\}} d^*(v) = 3$ .

Case 3ii a.  $x > 1$ .

Case 3ii a1.  $n > 6$ . Let  $v_1$  be a vertex of degree  $\Delta(G) - 2$  ( $v_1$  exists, since  $x > 1$ ).

Consider the graph  $G \setminus \{a,b\}$ . By Lemma 2.1, a vertex  $v$  of degree  $\Delta - 2$  in  $G$  is only joined in  $G$  to vertices  $w$  such that  $d_G^*(w) > \Delta - (\Delta - 2) + 1 = 3$ . Therefore no vertex joined to  $a$  has degree  $\Delta - 2$ . Therefore  $\delta(G \setminus \{a,b\}) \geq \Delta - 3$ . Also  $|V(G \setminus \{a,b\})| = 2n - 1$ . Since  $|V(G)|$  is odd and  $x \geq 1$ , then, by Lemma 7.3,

$$\Delta(G) > \left\lceil \frac{2(|V(G)| - 1)}{3} \right\rceil$$

and it is easy to verify that  $\Delta(G) - 3 \geq n$ , since  $n \geq 6$ .

Therefore  $\delta(G \setminus \{a,b\}) \geq \frac{1}{2}|V(G) \setminus \{a,b\}|$ , so  $G \setminus \{a,b\}$  contains a Hamiltonian cycle. Then  $G$  contains a near 1-factor  $F$  including  $ab$ , but including no edge incident with  $v_1$ .

Then  $G \setminus F$  has the same four vertices of maximum degree and  $d_{G \setminus F}^*(a) = 1$ ; we may assume that  $a$  is joined to  $c$ . Then, by Lemma 2.4,  $G \setminus F$  and  $G \setminus (F \cup ac)$  have the same Class. But  $G \setminus (F \cup ac)$  has just two vertices of maximum degree, so, by Lemma 2.5, is Class 1. Working back, it

follows that  $G$  is Class 1.

Case 3ii a2.  $n = 5$  or  $6$ . We shall show that  $G \setminus \{a, b\}$  has a Hamiltonian cycle. It then follows by the argument used in Case 3ii a1 that  $G$  is Class 1.

If  $v$  is a vertex such that  $d_G(v) = \Delta(G) - 2$  and if  $w$  is adjacent to  $v$  in  $G$ , then, by Lemma 2.1,  $d_G^*(w) \geq \Delta(G) - d_G(v) + 1 = \Delta - (\Delta - 2) + 1 = 3$ . For any vertex  $v$  such that  $d_{G \setminus \{a, b\}}(v) = \Delta(G) - 3$ , it follows that  $d_G(v) = \Delta(G) - 1$  or  $\Delta(G) - 2$ ; if  $d_G(v) = \Delta(G) - 1$  then  $v$  is joined in  $G$  to  $a$  and  $b$ , and so, in particular, is joined to  $b$ ; and if  $d_G(v) = \Delta(G) - 2$  then  $v$  is joined in  $G$  either to  $a$  or to  $b$ ; but  $v$  cannot be joined to  $a$  since  $d^*(a) = 2$ ; therefore  $v$  is joined to  $b$ . Thus, to summarize, if  $d_{G \setminus \{a, b\}}(v) = \Delta(G) - 3$ , then  $v$  is joined to  $b$ . Therefore there are at most  $\Delta(G) - 3$  vertices of degree  $\Delta - 3$  in  $G \setminus \{a, b\}$ .

Consider the case when  $n = 5$ . By Lemma 7.3,  $\Delta(G) \geq \left\lceil \frac{20}{3} \right\rceil = 7$ , and, as in Case 3ii a1,  $\delta(G \setminus \{a, b\}) \geq \Delta(G) - 3$ . If  $\Delta(G) \geq 8$ , then  $\delta(G \setminus \{a, b\}) \geq 5$ , so  $\delta(G \setminus \{a, b\}) \geq \frac{1}{2}|V(G \setminus \{a, b\})|$ , and so, by Lemma 7.5,  $G \setminus \{a, b\}$  has a Hamiltonian cycle. If  $\Delta(G) = 7$  then  $\delta(G \setminus \{a, b\}) \geq 4$ . But, as shown above, there are at most  $\Delta - 3 = 4$  vertices of degree 4. Therefore, if  $d_1 \leq \dots \leq d_9$  is the degree sequence of  $G \setminus \{a, b\}$ , then  $d_5 \geq 5$ , and it follows from Lemma 7.4 that  $G \setminus \{a, b\}$  has a Hamiltonian cycle.

Now consider the case when  $n = 6$ . Then  $\Delta(G) \geq 8$  and  $\delta(G \setminus \{a, b\}) \geq \Delta(G) - 3$ . If  $\Delta(G) = 9$  then  $\delta(G \setminus \{a, b\}) \geq 6$  and so  $\delta(G \setminus \{a, b\}) \geq \frac{1}{2}|V(G \setminus \{a, b\})|$ , and so  $G$  has a Hamiltonian cycle. If  $\Delta(G) = 8$  then  $\delta(G \setminus \{a, b\}) \geq 5$ . But there are at most 5 vertices in  $G \setminus \{a, b\}$  of degree 5. Therefore, if  $d_1 \leq \dots \leq d_{11}$  is the degree sequence of  $G \setminus \{a, b\}$ , then  $d_6 \geq 6$ , and it follows from Lemma 7.4 that  $G$  has a Hamiltonian cycle.

Case 3ii b.  $x = 0$ .

Case 3ii b1.  $\Delta(G) \geq n+3$ . We show that if  $G$  contains a near 1-factor  $F$  such that  $ab$  is in  $F$  and no edge incident with  $w$ , where  $w$  is a vertex of degree  $\Delta-1$  not adjacent to  $a$ , is in  $F$ , then  $G$  is Class 1. There is a suitable vertex  $w$  since  $d(a) = \Delta \leq 2n-2$ . The graph  $G \setminus F$  has five vertices  $a, b, c, d$  and  $w$ , of maximum degree, but  $d^*(a)=1$  so, by Lemma 2.4,  $G \setminus F$  and  $(G \setminus F) \setminus \{a\}$  have the same Class. The graph  $(G \setminus F) \setminus \{a\}$  has three vertices of maximum degree  $\Delta(G)-1$ . Since  $\delta((G \setminus F) \setminus \{a\}) > n$ , it follows that  $(G \setminus F) \setminus \{a\}$  is connected. Therefore, by the proposition,  $(G \setminus F) \setminus \{a\}$  is Class 1, since it has even order. Working back, it follows that  $G$  is Class 1.

We now use Dirac's condition (Lemma 7.5) to show that  $F$  exists. The graph  $G \setminus \{a, b\}$  has minimum degree  $\Delta(G)-3$  and has  $2n-1$  vertices. Since  $\Delta(G) \geq n+3$ ,  $\Delta(G)-3 \geq n = \left\lceil \frac{2n-1}{2} \right\rceil = \left\lceil \frac{1}{2} |V(G \setminus \{a, b\})| \right\rceil$ . Therefore  $G \setminus \{a, b\}$  has a Hamiltonian cycle, and consequently  $G$  has a suitable near 1-factor  $F$ .

Case 3ii b2.  $\Delta(G)=n+2$ . The graph  $G \setminus \{a, b\}$  has at most  $n-1$  vertices of degree  $n-1$  and no vertices of lower degree. Lemma 7.4 applies and shows that  $G \setminus \{a, b\}$  has a Hamiltonian cycle. The remainder of the argument is the same as in the case immediately above.

Case 3ii b3.  $\Delta(G)=n+1$ . The vertices of degree  $\Delta-1$  are all joined to exactly two vertices of degree  $\Delta$ . Hence  $G \setminus \{a, b, c, d\}$  has  $2n-3$  vertices and is regular of degree  $n-2$ . This graph is 2-connected and hence by Jackson's theorem (Lemma 7.6), since  $3(n-2) \geq 2n-3$  (as  $n \geq 3$ ),  $G \setminus \{a, b, c, d\}$  has a Hamiltonian cycle. We take alternate edges of this cycle avoiding a vertex  $w$  not joined to  $a$ . Clearly  $w$  exists since  $d(a) \leq n+1$ . We take two more edges (here we assume  $a$  is joined to  $c$  as well as to  $b$ ): either  $ab, cd$ , if  $w$  is not joined to  $b$ , or  $ac, bd$ , if  $w$  is joined to  $b$ .

Then in either case  $d^*(a)=1$  and so, by Lemma 2.3,

either  $G \setminus (F \cup ac)$  has the same Class as  $G \setminus F$

or  $G \setminus (F \cup ab)$  has the same Class as  $G \setminus F$ .

In either case we have at most 3 vertices of maximum degree not all joined to each other and so, by the proposition, we have a Class 1 graph. Hence, working back,  $G$  is Class 1.

Case 3ii b4.  $\Delta(G) \leq n$ . Since each vertex is joined to at least two of  $\{a,b,c,d\}$ , by counting the edges between  $\{a,b,c,d\}$  and  $V(G) \setminus \{a,b,c,d\}$ , it follows that

$$2(2n-3) \leq 2(\Delta-2) + 2(\Delta-3)$$

so that  $n+1 \leq \Delta$ . Therefore this case does not arise.

Case 3iii.  $d^*(a) = 3$ .

Case 3iii a.  $x \geq 1$ .

Case 3iii a1.  $\Delta(G) \geq n+4$ . Let  $v$  be a vertex of degree  $\Delta(G)-2$  and suppose that  $v$  is joined to  $a$  and  $d$ . Let  $w$  be a vertex not joined to  $a$ . If we can find a near 1-factor  $F$  which includes the edge  $ad$  but does not include any edge incident with  $w$ , then it follows that  $G$  is Class 1; the argument to show this is as follows:

In the graph  $G \setminus F$  there are at most five vertices,  $a,b,c,d,w$ , of maximum degree  $\Delta(G)-1$ . Clearly  $d^*_{G \setminus F}(a)=2$  ( $a$  is now not joined to  $w$  or  $d$ ) and  $d_{G \setminus F}(v)=\Delta(G)-3$ . Let  $H$  be a critical subgraph of  $G \setminus F$  with the same maximum degree  $\Delta(G)-1$ . It follows that, if  $a \in V(H)$  then  $d^*_H(a) \leq 2$ , and if  $v \in V(H)$  then  $d_H(v) \in \Delta(G)-3$ . But, if  $\forall a \in E(H)$  then, by Lemma 2.1,  $d^*_H(a) \geq \Delta(H) - d_H(v) + 1 \geq (\Delta(G)-1) - (\Delta(G)-3) + 1 = 3$ .

Therefore  $va \notin E(H)$ . Therefore  $G \setminus F$  and  $G \setminus (F \cup \{a\})$  have the same Class.

In the graph  $G \setminus (F \cup \{va\})$  there are at most four vertices,  $b, c, d, w$ , of maximum degree. Clearly  $d_{G \setminus (F \cup \{va\})}^*(d) = 3$  and  $d_{G \setminus (F \cup \{va\})}^*(v) = \Delta(G) - 4$ . It follows that if  $d \in V(H)$  then  $d_H^*(d) \leq 3$  and if  $v \in V(H)$  then  $d_H(v) \leq \Delta(G) - 4$ . But, if  $vd \in E(H)$ , then, by Lemma 2.1,  $d_H^*(d) \geq \Delta(H) - d_H(v) + 1 \geq (\Delta(G) - 1) + (\Delta(G) - 4) + 1 = 4$ . Therefore  $vd \notin E(H)$ . Therefore  $G \setminus (F \cup \{va\})$  and  $G \setminus (F \cup \{va, vd\})$  have the same Class.

The graph  $G \setminus (F \cup \{va, vd\})$  has three vertices of maximum degree; all vertices except  $v$  have degree at least  $\Delta(G) - 3 \geq n + 3 - 3 = n$ , so, since  $n \geq 5$ ,  $G \setminus (F \cup \{va, vd\})$  is connected. However, in this graph, the degree of  $v$  is three less than the maximum degree, so, by the proposition,  $G \setminus (F \cup \{va, vd\})$  is Class 1. Therefore  $G$  is Class 1.

Since  $\Delta(G) \geq n + 4$ , it follows that  $\delta(G \setminus \{a, d\}) \geq n = \left\lceil \frac{1}{2} |V(G \setminus \{a, d\})| \right\rceil$ , so, by Lemma 7.5,  $G \setminus \{a, d\}$  has a Hamiltonian cycle, and, consequently,  $G$  has a near 1-factor containing  $ab$ , but containing no vertex incident with  $w$ .

Case 3iii a2.  $\Delta(G) = n + 3$ . Suppose first that there are two vertices  $v, w$  both of degree  $\Delta(G) - 2$  and that two of  $\{a, b, c, d\}$ , say  $a$  and  $d$ , are joined to both  $v$  and  $w$ . Let  $d_1 \leq d_2 \leq \dots \leq d_{2n}$  be the degree sequence of  $G \setminus \{v\}$ . Since  $v$  is joined to at most  $n - 1$  vertices of degree  $n + 1$ , it follows that  $d_n \geq n + 1$ . One may easily verify that therefore  $G \setminus \{v\}$  satisfies the conditions of Lemma 7.4 for the existence of a Hamiltonian cycle containing a specified edge. Let  $F$  be a 1-factor of  $G \setminus \{v\}$  containing the edge  $ad$  obtained by deleting alternate edges of a Hamiltonian cycle containing  $ad$ .

Let  $H$  be a critical subgraph of  $G \setminus F$  with the same maximum degree  $\Delta(G) - 1$ . If  $aw \in E(H)$  then it would follow from Lemma 2.1 that

$$d_H^*(a) \geq \Delta(H) - d_H(w) + 1 \geq (\Delta(G) - 1) - (\Delta(G) - 3) + 1 = 3.$$

However this is impossible since  $d_{G \setminus F_1}^*(a) = 2$ . Therefore  $aw \notin E(H)$ . Similarly  $dw \notin E(H)$ . Consequently  $H$  has at most two vertices,  $b, c$ , of maximum degree; however, it then follows by Lemma 2.5 that  $H$  is Class 1. Therefore  $G \setminus F$  is Class 1, and so  $G$  is Class 1.

If there do not exist two such vertices, then we can use the argument of Case 3iii a1. By Lemma 7.4,  $G \setminus \{a, d\}$  has a Hamiltonian cycle, since, if  $d_1 \leq \dots \leq d_{2n-1}$  is the degree sequence of  $G \setminus \{a, d\}$ , then  $d_2 \geq n$  and so  $d_n \geq n$ .

Case 3iii a3.  $\Delta(G) \leq n + 2$ . By Lemma 7.3,

$$\frac{4n}{3} \leq \Delta \leq n + 2,$$

and this implies that  $n \leq 6$ . There are only two possibilities to consider:  $n = 6$  and  $\Delta = 8$ ;  $n = 5$  and  $\Delta = 7$ . We shall show that in neither case does a critical graph exist.

Consider first the case when  $n = 6$  and  $\Delta(G) = 8$ . Suppose a critical graph  $G$  exists. A vertex of degree  $6 = \Delta - 2$  is joined to either all four of  $\{a, b, c, d\}$  and to two other vertices, or to at least three other vertices. As explained in Case 3ii a1 of the proof of Theorem 7.1 in Section 7.4, there are at least three edges joining any vertex adjacent to a vertex of degree  $\Delta - 2$  to  $\{a, b, c, d\}$ , and so, in both of the above cases, there are three vertices other than  $\{a, b, c, d\}$  each joined to at least three of  $\{a, b, c, d\}$ . The remaining six vertices of  $V(G) \setminus \{a, b, c, d\}$  are, by Lemma 2.2, joined to at least two of  $\{a, b, c, d\}$ . Therefore there are at least  $3 \cdot 3 + 6 \cdot 2 = 21$  edges from



$V(G) \setminus \{a, b, c, d\}$  to  $\{a, b, c, d\}$ . However, since each of  $\{a, b, c, d\}$  has degree 8, from each of  $\{a, b, c, d\}$  there are five edges to  $V(G) \setminus \{a, b, c, d\}$ , and thus 20 such edges in all, a contradiction. Therefore there is no such critical graph.

The argument in the case when  $n = 5$  and  $\Delta = 7$  is similar, but slightly more involved. If a vertex of degree 5 is joined to all four of  $\{a, b, c, d\}$ , then there are at least  $5 \cdot 2 + 3 + 4 = 17$  edges from  $V(G) \setminus \{a, b, c, d\}$  to  $\{a, b, c, d\}$ . If there are three edges from the vertex of degree 5 to  $\{a, b, c, d\}$ , then there are also three such edges from at least two other vertices of  $V(G) \setminus \{a, b, c, d\}$ ; if there are only two edges from the vertex of degree 5 to  $\{a, b, c, d\}$ , then there are three vertices of  $V(G) \setminus \{a, b, c, d\}$ , each having at least three such edges on them. There are therefore altogether at least  $4 \cdot 2 + 3 \cdot 3 = 17$ , again, such edges. However, by the other argument, there are only 16 such edges, a contradiction.

Case 3iii b.  $x = 0$ .

Case 3iii b1.  $\Delta(G) \geq \frac{3|V(G)|}{4}$  and  $\Delta(G) \geq n+4$ . Since  $\Delta(G) \leq 2n-2$  there are two vertices, say  $v_1$  and  $v_2$ , which are both non-adjacent to  $a$ . We can assume  $v_1$  is adjacent to  $d$ .

Consider the graph  $G \setminus \{a, b, d, v_1\}$ . We have  $\delta(G \setminus \{a, b, d, v_1\}) \geq \Delta(G) - 5$  and  $|V(G \setminus \{a, b, d, v_1\})| = 2n - 3$ . Since  $\Delta(G) - 5 \geq n - 1$  we have  $\delta(G \setminus \{a, b, d, v_1\}) \geq \frac{1}{2}|V(G \setminus \{a, b, d, v_1\})|$ , so, by Lemma 7.5,  $G \setminus \{a, b, d, v_1\}$  contains a Hamiltonian cycle. Therefore  $G$  contains a near 1-factor  $F_1$  containing the edges  $ab$  and  $dv_1$ , but containing no edge incident with  $v_2$ .

The graph  $G \setminus F_1$  contains five vertices  $a, b, c, d, v_2$ , of maximum degree  $\Delta(G)-1$ . Now consider the graph  $(G \setminus \{a, d\}) \setminus F_1$ . We have  $\delta((G \setminus \{a, d\}) \setminus F_1) \geq \Delta(G)-4$  and  $V((G \setminus \{a, d\}) \setminus F_1) = 2n-1$ . If  $\Delta(G)-4 \geq n$  then  $\delta((G \setminus \{a, d\}) \setminus F_1) \geq \frac{1}{2}|V((G \setminus \{a, d\}) \setminus F_1)|$  so, by Lemma 7.5,  $(G \setminus \{a, d\}) \setminus F_1$  contains a Hamiltonian circuit. Therefore  $G \setminus F_1$  contains a near 1-factor  $F_2$  containing  $ad$  but containing no edge incident with  $v_1$ .

The graph  $G \setminus (F_1 \cup F_2)$  has six vertices,  $a, b, c, d, v_1, v_2$ , of maximum degree  $\Delta(G)-2$ , and of these  $c$  is the only one adjacent to  $a$ . By Lemma 2.4,  $G \setminus (F_1 \cup F_2)$  and  $(G \setminus a) \setminus (F_1 \cup F_2)$  have the same Class. Now  $(G \setminus a) \setminus (F_1 \cup F_2)$  has four vertices,  $b, d, v_1, v_2$  of maximum degree  $\Delta(G)-2$  and  $d$  is non-adjacent to  $v_1$ ; it has  $\Delta(G)-3$  vertices of degree  $\Delta(G)-4$  and therefore  $2n - (\Delta(G)-3) = 2n - \Delta(G) + 3$  vertices of degree at least  $\Delta(G)-3$ .

If the graph  $(G \setminus a) \setminus (F_1 \cup F_2)$  is Class 2, then it contains a critical subgraph  $G^*$  with the same maximum degree  $\Delta(G)-2$ . By Theorem 7.1,  $|V(G^*)|$  is not even. Let  $|V(G^*)| = 2n^*+1$  for some  $n^* \leq n$ . Let  $G^*$  have  $r^*$  vertices of maximum degree. Then  $r^* \leq r = 4$ , and, by Lemma 2.5,  $3 \leq r^*$ . If  $r = 3$  then, by the proposition,  $G^*$  has  $\Delta(G^*)+1$  vertices of degree at least  $\Delta(G^*)-1$ , so  $(G \setminus a) \setminus (F_1 \cup F_2)$  has at least  $\Delta(G)-1$  vertices of degree at least  $\Delta(G)-3$ . Since the implication (i)  $\Rightarrow$  (ii) has been established when  $r^* = 4$  in Cases 3i and 3ii, it follows that, if  $r^* = 4$ , then  $|E(G^*)| \geq n^*\Delta(G)+1$ , and so, by Lemma 2.7,  $|E(G^*)| = n^*\Delta(G^*)+1$ . If  $r = 4$ , then it follows that either  $G^*$  has four vertices of degree  $2n^*-1$  ( $=\Delta(G)-2$ ) and the rest  $(\Delta(G)-4)$  have degree  $2n^*-2$  ( $=\Delta(G)-3$ ) so that  $(G \setminus a) \setminus (F_1 \cup F_2)$  has at least  $\Delta(G)$  vertices of degree  $\geq \Delta(G)-3$ , or  $G$  has four vertices of degree  $2n^*$  ( $=\Delta(G)-2$ ), one of degree  $2n^*-2$  and the rest  $(\Delta(G)-6)$

have degree  $2n-1$  ( $= \Delta(G)-3$ ), so that  $(G \setminus a) \setminus (F_1 \cup F_2)$  has at least  $\Delta(G)-2$  vertices of degree at least  $\Delta(G)-3$ .

Since  $(G \setminus a) \setminus (F_1 \cup F_2)$  has  $2n-\Delta(G)+3$  vertices of degree at least  $\Delta(G)-3$ , it follows in the first and third cases that

$$\Delta(G)-1 \leq 2n-\Delta(G)+3,$$

so that  $n+2 \geq \Delta$ , and in the middle case it follows similarly that  $n+1 \geq \Delta(G)$ . But in both cases this contradicts our assumption here that  $\Delta \geq n+4$ . Therefore  $(G \setminus a) \setminus (F_1 \cup F_2)$  is Class 1. Working back, it follows that  $G$  is Class 1.

Case 3iii b2.  $\frac{3}{4}|V(G)| > \Delta(G) \geq n+4$ . If  $d^*(v) \geq 3$  for all  $v \in V(G)$ , then, counting edges, it is easy to see that  $\Delta(G) \geq \frac{3}{4}|V(G)|$ . Therefore, in this case, there is a vertex, say  $v_1$ , such that  $d^*(v_1) = 2$ . Suppose  $v_1$  is non-adjacent to both  $a$  and  $d$ . Since  $\Delta(G) \leq 2n-2$ , there is a vertex  $v_2 \neq v_1$  which also is non-adjacent to  $a$ .

Consider the graph  $G \setminus \{a, b\}$ . We have  $\delta(G \setminus \{a, b\}) \geq \Delta(G)-3$  and  $|V(G \setminus \{a, b\})| = 2n-1$ . Since  $\Delta(G)-3 \geq n$  we have  $\delta(G \setminus \{a, b\}) \geq \frac{1}{2}|V(G \setminus \{a, b\})|$ , so, by Lemma 7.5,  $G \setminus \{a, b\}$  contains a Hamiltonian cycle. Therefore  $G$  contains a near 1-factor  $F_1$  which contains the edge  $ab$  but contains no edge incident with  $v_1$ .

The graph  $G \setminus F_1$  contains five vertices,  $a, b, c, d, v_1$ , of maximum degree  $\Delta(G)-1$ . Now consider the graph  $(G \setminus \{a, d\}) \setminus F_1$ . We have  $\delta((G \setminus \{a, d\}) \setminus F_1) \geq \Delta(G)-4$  and  $|V((G \setminus \{a, d\}) \setminus F_1)| = 2n-1$ . Since  $\Delta(G)-4 \geq n$  it follows that  $\delta((G \setminus \{a, d\}) \setminus F_1) \geq \frac{1}{2}|V((G \setminus \{a, d\}) \setminus F_1)|$  so, by Lemma 7.5,  $(G \setminus \{a, d\}) \setminus F_1$  contains a Hamiltonian cycle. Therefore  $G \setminus F_1$  contains a near 1-factor  $F_2$  which contains the edge  $ad$ , but contains no edge incident

with  $v_2$ .

The graph  $G \setminus (F_1 \cup F_2)$  has six vertices,  $a, b, c, d, v_1, v_2$ , of maximum degree  $\Delta(G) - 2$ , and of these  $c$  is the only one adjacent to  $a$ . The argument now proceeds exactly as in the previous case, and it follows that  $G$  is Class 1.

Case 3iii b3.  $\Delta(G) = n+3$ . As in the previous case, we may take  $v_1$  to be a vertex non-adjacent to  $a$  and  $b$ , and  $v_2 (\neq v_1)$  to be a vertex non-adjacent to  $a$ . Again we show that  $G$  contains two edge-disjoint near 1-factors  $F_1$  and  $F_2$ , where  $F_1$  contains the edge  $ab$  but no edge incident with  $v_1$ , and  $F_2$  contains the edge  $ad$  but no edge incident with  $v_2$ . It then follows that  $G$  is Class 1.

There are altogether  $4n$  edges joining  $V(G) \setminus \{a, b, c, d\}$  to  $\{a, b, c, d\}$ . Of these,  $2(2n-3) = 4n-6$  are accounted for by the fact that  $d_G^*(v) \geq 2$  for each  $v \in V(G)$ . Therefore

$$G \setminus \{a, b, c, d\} \cong (n-2)^a (n-1)^{6-2a} n^{2n+a-9},$$

since  $(2n-3) - (6-2a) - a = 2n+a-9$ , where  $a \leq 3$ .

If there is no edge  $v_1 v_2$  in  $G \setminus \{a, b, c, d\}$ , insert it to form a graph  $G^*$ ; otherwise let  $G^* = G \setminus \{a, b, c, d\}$ . Let  $d_1 \leq d_2 \leq \dots \leq d_{2n-3}$  be the degree sequence of  $G \setminus \{a, b, c, d\}$ . We shall apply Lemma 7.4 to show that  $G^*$  has a Hamiltonian cycle containing  $v_1 v_2$ . In this case, this follows if we show that if  $d_{n-3} = n-2$  then  $d_{n-1} \geq n$ . If  $d_{n-3} > n-2$  then  $G^*$  has such a Hamiltonian cycle. Suppose therefore that  $d_{n-3} = n-2$ . Then  $n-3 \leq a \leq 3$ , and so  $n \leq 6$ . If  $n = 6$  then the degree sequence is  $(4, 4, 4, 6, 6, 6, 6, 6, 6)$ , so  $d_{n-1} = n$ . If  $n = 5$  then  $G \setminus \{a, b, c, d\} = 3^a 4^{6-2a} 5^{1+a}$ , so  $G \setminus \{a, b, c, d\}$  has  $1+2a$  vertices of odd degree, which is impossible. Therefore  $G^*$  contains a Hamiltonian cycle containing  $v_1 v_2$ .

Therefore  $G$  contains the two near 1-factors  $F_1$  and  $F_2$  with the desired properties.

Case 3iii b4.  $\Delta(G) = n+2$ . There are  $4(n-1)$  edges from  $V(G) \setminus \{a,b,c,d\}$  to  $\{a,b,c,d\}$ . By Lemma 2.2, since  $|V(G) \setminus \{a,b,c,d\}| = 2n-3$ , either (A) there is one vertex  $w_0 \in V(G) \setminus \{a,b,c,d\}$  such that  $d^*(w_0) = 4$  and  $d^*(v) = 2$  for  $v \in V(G) \setminus \{a,b,c,d,w_0\}$ , or (B) there are two vertices  $w_1$  and  $w_2$  such that  $d^*(w_1) = d^*(w_2) = 3$  and  $d^*(v) = 2$  for  $v \in V(G) \setminus \{a,b,c,d,w_1,w_2\}$ .

Suppose that  $G \setminus \{a,b,c,d\}$  contains a Hamiltonian cycle with two consecutive vertices  $v_1$  and  $v_2$  with the properties that there is a vertex in  $\{a,b,c,d\}$  which is not adjacent to either of  $v_1$  and  $v_2$ , and that  $d^*(v_1) = 2$  and  $d^*(v_2) \leq 3$ ; we may suppose that  $a$  is non-adjacent to both  $v_1$  and  $v_2$ , and that  $d$  is non-adjacent to  $v_1$ . Then  $G \setminus \{a,b,c,d\}$  contains two edge-disjoint near 1-factors  $F_1^*$  and  $F_2^*$ , such that  $F_1^*$  has no edge incident with  $v_1$  and  $F_2^*$  has no edge incident with  $v_2$ . It follows that  $G$  has two edge-disjoint near 1-factors  $F_1$  and  $F_2$ , where  $F_1 = F_1^* \cup \{ab, cd\}$  and  $F_2 = F_2^* \cup \{ad, bc\}$ . The graph  $G \setminus (F_1 \cup F_2)$  has six vertices,  $a, b, c, d, v_1, v_2$  of maximum degree, but of these  $c$  is the only one adjacent to  $a$ . Therefore, by Lemma 2.4,  $G \setminus (F_1 \cup F_2)$  and  $(G \setminus a) \setminus (F_1 \cup F_2)$  have the same Class. But  $(G \setminus a) \setminus (F_1 \cup F_2)$  has four vertices,  $b, d, v_1, v_2$ , of maximum degree, and  $d$  and  $v_1$  are non-adjacent.

We can now adapt the final part of the argument of Case 3iii b1 (when  $\Delta(G) < n+3$ , that argument does not all apply as it stands). From that argument it can be seen that two possibilities remain. One is that  $(G \setminus a) \setminus (F_1 \cup F_2)$  contains a critical subgraph  $G^*$  of order  $n+1$  with three vertices of maximum degree  $n$  and  $n-2$  vertices of degree  $n-1$ . The other is that  $(G \setminus a) \setminus (F_1 \cup F_2)$  contains a critical subgraph  $G^*$  of order  $n+1$  with four vertices of degree  $n$ ,  $n-4$  vertices of degree  $n-1$ , and one of degree  $n-2$ .

We know that  $(G \setminus a) \setminus (F_1 \cup F_2)$  has four vertices of degree  $\Delta(G) - 2 = n$ ,  $n-3$  vertices of degree  $n-1$  and  $n-1$  vertices of degree  $n-2$ . However in both cases it is easy to see that it is not possible to extend  $G^*$  to a graph with these parameters. Therefore  $(G \setminus a) \setminus (F_1 \cup F_2)$  is Class 1. Working back it follows that  $G$  is Class 1.

We now show that there always is such a Hamiltonian cycle. The degree sequence of  $G \setminus \{a, b, c, d\}$  is  $(n-3, n-1, n-1, \dots, n-1)$  in Case A and  $(n-2, n-2, n-1, n-1, \dots, n-1)$  in Case B;  $|V(G \setminus \{a, b, c, d\})| = 2n-3$ . By Lemma 7.4, in both cases,  $G \setminus \{a, b, c, d\}$  has a Hamiltonian cycle  $H$  with a prescribed edge  $v_1 v_2$ .

In Case A,  $H$  will have the required property unless the vertices going round  $H$  starting at  $w_0$  are joined to those of  $\{a, b, c, d\}$  indicated:

$$\{\alpha, \beta, \gamma, \delta\}, \{\alpha, \beta\}, \{\gamma, \delta\}, \dots, \{\alpha, \beta\}, \{\gamma, \delta\};$$

(here  $(\alpha, \beta, \gamma, \delta)$  is some permutation of  $(a, b, c, d)$ ).

The number of times the pairs  $\{\alpha, \beta\}$  and  $\{\gamma, \delta\}$  occur in this list is  $n-1$  (including each pair as a subset of  $\{\alpha, \beta, \gamma, \delta\}$ ). The degree of  $w_0$  in  $G \setminus \{a, b, c, d\}$  is  $n-3$ , so there is one vertex, say  $w^*$ , joined to the pair  $\{\alpha, \beta\}$ , which is not joined to  $w_0$ . There are only  $n-2$  vertices other than  $w_0$  which are joined to the pair  $\gamma, \delta$ , so  $w^*$  must be joined to a vertex  $w^{**} \neq w_0$  which is also joined to  $\alpha, \beta$ . By Lemma 7.4,  $G \setminus \{a, b, c, d\}$  has a Hamiltonian cycle which includes the edge  $w^* w^{**}$ . This is the required Hamiltonian cycle.

In case B,  $H$  will similarly have the required property unless the vertices going round  $H$  starting at  $w_1$  are joined to those of  $\{a, b, c, d\}$  indicated:

$$\{\alpha, \beta, \gamma\}, \{\gamma, \delta\}, \{\alpha, \beta\}, \dots, \{\gamma, \delta\}, \{\alpha, \beta\}, \{\beta, \gamma, \delta\}, \{\alpha, \delta\}, \{\beta, \gamma\}, \dots, \{\beta, \gamma\}, \{\alpha, \delta\};$$

(here again,  $(\alpha, \beta, \gamma, \delta)$  is some permutation of  $(a, b, c, d)$ ).

The number of vertices joined to both  $\alpha$  and  $\delta$  is at most  $n-2$ . If there is a vertex  $w^*$  adjacent to  $\beta$  and to  $\gamma$  and not adjacent to  $\alpha$  or  $\delta$ , then it must be adjacent to at least one vertex  $w^{**}$  which itself is not adjacent to both  $\alpha$  and  $\delta$ . If there is no such vertex  $w^*$  then the sequence above is the special case:

$$\{\alpha, \beta, \gamma\}, \{\alpha, \delta\}, \{\beta, \gamma, \delta\}, \{\alpha, \beta\}, \{\gamma, \delta\}, \dots, \{\alpha, \beta\}, \{\gamma, \delta\}.$$

There are  $n-2$  vertices adjacent to both  $\gamma$  and  $\delta$ . Therefore, a vertex  $w'$  adjacent to  $\alpha$  and  $\beta$  and not adjacent to  $\gamma$  and  $\delta$  is adjacent to at least one vertex  $w''$ , itself not adjacent to both  $\gamma$  and  $\delta$ . By Lemma 7.4,  $G \setminus \{a, b, c, d\}$  has a Hamiltonian cycle which includes the edge  $w^* w^{**}$  (or the edge  $w' w''$ ). This is the required Hamiltonian cycle.

Case 3iii b5.  $\Delta(G) = n+1$ . This case cannot arise, as the number of edges from  $\{a, b, c, d\}$  to  $G \setminus \{a, b, c, d\}$  would have to be  $4(n-2) = 4n-8$ , whereas, by Lemma 2.2, it must be at least  $2(2n-3) = 4n-6$ .

We have now proved the implication (i)  $\Rightarrow$  (ii) in Case 3.

The implication (ii)  $\Rightarrow$  (i) in Case 3. Suppose that  $|E(G)| = n\Delta(G) + 1$ .

Then  $G$  is Class 2. By Lemma 7.1,  $\Delta(G) \geq 2n-1$ . The graph  $G$  must contain a critical subgraph  $G^*$  of the same maximum degree. By Lemma 2.5,  $G^*$  has either three or four vertices of maximum degree.

If  $G^*$  has three vertices of maximum degree, then by the proposition,  $|V(G^*)| - 1 = \Delta(G^*) = \Delta(G) = |V(G)| - 1$ ; by the implication (i)  $\Rightarrow$  (ii) when  $r = 3$ ,  $|E(G^*)| = n\Delta(G^*) + 1$ . Therefore  $|E(G^*)| = n\Delta(G) + 1 = |E(G)|$ , so  $G = G^*$ ; but this contradicts the fact that  $G$  has four vertices of maximum degree.

Now suppose that  $G^*$  has four vertices of maximum degree. By Theorem 7.1,  $|V(G^*)| \neq 2n$ . Therefore  $|V(G^*)| = 2n+1$ . By the implication (i) $\Rightarrow$ (ii) of Theorem 7.2,  $|E(G^*)| = n\Delta(G^*)+1 = n\Delta(G)+1 = |E(G)|$ . It follows that  $G = G^*$ , and therefore  $G$  is critical.

The implication (i) $\Rightarrow$ (iii) in Case 3. This is obvious.

The implication (iii) $\Rightarrow$ (i) in Case 3. Suppose that  $G$  is 2-edge-connected and Class 2. Let  $G^*$  be a critical subgraph of  $G$  of the same maximum degree. By Lemma 2.5,  $G^*$  has either three or four vertices of maximum degree.

If  $G^*$  has three vertices of maximum degree then, by the proposition,  $\delta(G^*) = \Delta(G^*)-1$ . As  $G$  has four vertices of maximum degree, it follows that  $G$  is not 2-edge-connected. But this contradicts our assumption.

Now suppose that  $G^*$  has four vertices of maximum degree. Then, by the implication (i) $\Rightarrow$ (ii) when  $r = 4$ , it follows that

$$|E(G^*)| = n^*\Delta(G^*)+1 = n^*\Delta(G)+1,$$

where  $|V(G^*)| = 2n^*+1$ . By Lemma 7.1,  $\Delta(G^*) \geq 2n^*-1$ . It is easy to verify by counting that if  $\Delta(G^*) = 2n^*-1$  then  $\delta(G^*) = 2n^*-2$ , and that if  $\Delta(G^*) = 2n^*$  then  $G^*$  has one vertex of degree  $2n^*-2$ , the remainder having degree at least  $2n^*-1$ . If  $n \neq n^*$ , then it would follow that  $G$  was not 2-edge-connected, a contradiction. Therefore  $n = n^*$ , so  $V(G) = V(G^*)$  and  $|E(G^*)| = n\Delta(G)+1$ . But no edges can be added to  $G^*$  without creating a further vertex of maximum degree. Therefore  $G = G^*$ .

This completes the proof of Theorem 7.2.



### 7.6. Proofs of Theorems 7.3, 7.4 and 7.5.

Proof of Theorem 7.4. Let  $G$  be a 2-edge-connected graph with  $|V(G)| = 2n$  and with four vertices of maximum degree, and suppose that  $G$  is Class 2. Let  $G^*$  be a critical subgraph of  $G$  with the same maximum degree. Then by Theorem 7.1  $|V(G^*)|$  is odd, equalling  $2n^* + 1$ , say, so  $2n^* + 1 < |V(G)|$ . By Lemma 2.5,  $G^*$  has either three or four vertices of maximum degree.

If  $G^*$  has three vertices of maximum degree, then, by the proposition,  $\delta(G^*) = \Delta(G^*) - 1$ . Since  $G$  has four vertices of maximum degree,  $G$  cannot be 2-edge-connected, a contradiction.

If  $G^*$  has four vertices of maximum degree, then  $|E(G^*)| = n^* \Delta(G^*) + 1$ . By Lemma 7.1,  $2n^* - 1 \leq \Delta(G^*)$ . It is easy to verify by counting that if  $\Delta(G^*) = 2n^* - 1$  then  $\delta(G^*) = 2n^* - 2$ , and that, if  $\Delta(G^*) = 2n^*$ , then  $G^*$  has one vertex of degree  $2n^* - 2$ , the remainder having degree at least  $2n^* - 1$ . Since  $G$  has four vertices of maximum degree, it follows that  $G$  cannot be 2-edge-connected, a contradiction.

It follows that  $G$  is Class 1, as required.

### Proof of Theorem 7.3.

Necessity. If  $G$  is Class 2, then, since (iii)  $\Rightarrow$  (ii) in Theorem 7.2, it follows that  $|E(G)| > n\Delta(G)$ .

Sufficiency. If  $|E(G)| > n\Delta(G)$ , then, by Lemma 2.7,  $G$  is Class 2.

Proof of Theorem 7.5

Sufficiency. In Cases (i) and (ii), the sufficiency follows from Lemma 2.7 applied to  $G$  and, in Case (iii), the sufficiency follows from Lemma 2.7 applied to  $C_2$ .

Necessity. Assume  $G$  is Class 2. Then  $G$  contains a critical subgraph  $G^*$  with the same maximum degree and three or four vertices of maximum degree. If  $G^*$  has three vertices of maximum degree then  $G^* \cong (2m-1)^{2m-2}(2m)^3$  for some  $m$ , by the proposition, so  $G \setminus G^*$  is joined to  $G^*$  by exactly one edge. If  $G^*$  has four vertices of maximum degree then  $G^* \cong (2m-2)^{2m-3}(2m-1)^4$  or  $G^* \cong (2m-2)(2m-1)^{2m-4}(2m)^4$  for some  $m$ , since, by Theorems 7.1 and 7.2,  $|E(G^*)| = \left\lfloor \frac{1}{2} |V(G^*)| \right\rfloor \Delta(G) + 1$  and so, by Lemma 7.1,  $\Delta \geq |V(G^*)| - 2$ . The case  $G^* \cong (2m-2)^{2m-3}(2m-1)^4$  with  $m < n$  is excluded since  $G$  is connected. If  $m < n$  and  $G^* \cong (2m-2)(2m-1)^{2m-4}(2m)^4$  there can only be one further edge of  $G$  incident with  $G^*$ , namely an edge incident with the vertex of degree  $2m-2$  in  $G^*$ .



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## 8. The chromatic class of graphs with many vertices of maximum degree

### 8.1 Introduction and summary of results

In this chapter and the previous chapter we obtain some results about the chromatic class of graphs with a fixed number ( $r$ ) of vertices of maximum degree. In the last chapter we proved each of the four conjectures in Chapter 7 for  $1 \leq r \leq 4$ , and in this chapter we prove these conjectures for general values of  $r$ , but we have to assume that  $\Delta(G)$  is large. We also describe all Class 2 graphs with  $r$  vertices of maximum degree and  $\Delta(G)$  large.

We obtain the following results:

Theorem 8.1. Let  $G$  have  $r$  vertices of maximum degree  $\Delta$ , and let

$|V(G)| = 2n$ . If  $\Delta(G) \geq n + \frac{5}{2}r - 4$ , then  $G$  is not critical.

Theorem 8.2. Let  $G$  have  $r$  vertices of maximum degree  $\Delta$ , and let

$|V(G)| = 2n+1$ . Let

$$\Delta \geq \begin{cases} n + \frac{7}{2}r - \frac{1}{4}t - \frac{11}{4} & \text{if } \Delta = 2n + 1 - r + t \text{ and } t > 0, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n + 2 - r. \end{cases}$$

Then conditions (i) - (iv) below are equivalent:

- (i)  $G$  is critical,
- (ii)  $|E(G)| = n\Delta + 1,$
- (iii)  $G$  is  $(r-2)$ -edge-connected and Class 2, and  $|E(G)| \leq n\Delta + 1,$
- (iv)  $\text{def}(G) = \Delta - 2.$

Each of the above conditions implies the following:

(v) The edge-connectivity  $\lambda(G)$ , satisfies  $\lambda(G) \geq 2n-r+2$ .

Note that there is no ambiguity when  $\Delta = 2n+2-r$ , as the two inequalities are identical then. Note also that the inequalities for  $\Delta$  can be rewritten:

$$\Delta \geq \begin{cases} \frac{6}{5}n + \frac{13}{5}r - 2 & \text{if } \Delta \geq 2n+2-r, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r. \end{cases}$$

Again there is no ambiguity when  $\Delta = 2n+2-r$ .

These two theorems, which give conditions for graphs to be critical, are applied to give the following two results on the chromatic class of graphs of sufficiently high degree and edge-connectivity.

Theorem 8.3. Let G have r vertices of maximum degree, let  $|V(G)| = 2n$  and let G have edge-connectivity at least  $(r-2)$ .

If

$$\Delta \geq \begin{cases} n + \frac{7}{2}r - \frac{1}{4}t - \frac{11}{4} & \text{if } \Delta = 2n+1-r+t \text{ and } t > 0, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r, \end{cases}$$

then G is Class 1.

Theorem 8.4. Let G have r vertices of maximum degree, let  $|V(G)| = 2n+1$  and let G be  $(r-2)$ -edge-connected. Let

$$\Delta \geq \begin{cases} n + \frac{7}{2}r - \frac{1}{4}t - \frac{11}{4} & \text{if } \Delta = 2n+1-r+t \text{ and } t > 0, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r. \end{cases}$$

Then

G is Class 2

if and only if

$$|E(G)| > n \Delta(G).$$

In the proof of Theorem 8.2 we need the following result, which is very similar in essence to Theorem 8.3 and 8.4 and is of interest in its own right. (We thank Dr. F. Holroyd for drawing our attention to this result.)

Theorem 8.5. Let  $G$  have  $r$  vertices of maximum degree and let

$$|V(G)| = 2n + 1. \quad \text{Let}$$

$$\delta(G) \geq \begin{cases} n + \frac{5}{2}r - \frac{1}{4}t - \frac{3}{4} & \text{if } \Delta = 2n+1-r+t \text{ and } t > 0, \\ n + \frac{5}{2}r - 1 & \text{if } \Delta \leq 2n+2-r. \end{cases}$$

Then  $G$  is Class 2

if and only if

$$|E(G)| > n\Delta(G).$$

By again considering the minimum degree, we have the following theorem from which we can deduce Theorem 8.1.

Theorem 8.6. Let  $G$  have  $r$  vertices of maximum degree and let

$$|V(G)| = 2n. \quad \text{If } \delta(G) \geq n + \frac{3}{2}r - 2, \text{ then } G \text{ is Class 1.}$$

All Class 2 graphs with  $r$  vertices of maximum degree, where the maximum degree is sufficiently high are described by Theorems 8.7 and 8.4.

Theorem 8.7. Let  $G$  have an edge-cut  $S$  with  $|S| < r - 2$ , let  $G$  have  $r$  vertices of maximum degree, and let  $\frac{1}{2}|V(G)| = n$ . Let

$$\Delta \geq \begin{cases} n + \frac{7}{2}r - \frac{1}{4}t - \frac{11}{4} & \text{if } \Delta = 2n+1-r+t \text{ and } t > 0, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r. \end{cases}$$

Then

$G$  is Class 2

if and only if

$S$  separates  $G$  into two subgraphs  $G_1$  and  $G_2$ , where

$$|V(G_1)| > |V(G_2)|, \quad |V(G_1)| \text{ is odd, and}$$

$$|E(G_1)| > \Delta(G) \left\lfloor \frac{|V(G_1)|}{2} \right\rfloor.$$

## 8.2. Proof of Theorems 8.1 and 8.6.

We first prove Theorem 8.6.

Proof of Theorem 8.6. Suppose that  $G$  has  $r$  vertices of maximum degree, has  $|V(G)| = 2n$  and satisfies  $\delta(G) \geq n + \frac{3}{2}r - 2$ .

Let  $G_r$  be the induced subgraph of  $G$  on the  $r$  vertices of maximum degree. Partition  $E(G_r)$  into  $r$  partial matchings,  $M_1, \dots, M_r$ , such that, for  $1 \leq i \leq r$ ,  $M_i$  is a maximal (by inclusion) matching in the graph  $G_r \setminus (M_1 \cup \dots \cup M_{i-1})$ . This can be done as follows:

Firstly  $G_r$  can be given a proper edge-colouring with  $r$  colours (by Vizing's theorem); let the  $i$ -th colour class be  $M_i^*$  ( $1 \leq i \leq r$ ).

For  $1 \leq i \leq r$ , define  $M_1, M_2, \dots, M_r$  in sequence so that  $M_i$  is a matching which contains  $M_i^* \setminus (M_1 \cup \dots \cup M_{i-1})$  and which is maximal in the graph  $G \setminus (M_1 \cup \dots \cup M_{i-1})$ .

Next let  $F_1, \dots, F_{r-1}$  be  $r-1$  edge-disjoint 1-factors of  $G$  such that  $M_i \subseteq F_i$  ( $1 \leq i \leq r-1$ ). We now show that such 1-factors do exist. Let  $1 \leq j \leq r-1$  and suppose that  $F_1, \dots, F_{j-1}$  exist and that  $(F_1 \cup \dots \cup F_{j-1}) \cap (M_j \cup \dots \cup M_r) = \emptyset$ ; we now show that  $F_j$  exists.

Let  $H_j = G \setminus (F_1 \cup \dots \cup F_{j-1})$ .

Then

$$\begin{aligned} \delta(H_j \setminus V(M_j)) &\geq \delta(G) - (j-1) - |V(M_j)| \\ &\geq \delta(G) - (j-1) - r. \end{aligned}$$

By Lemma 7.5, if

$$\delta(H_j \setminus V(M_j)) \geq \frac{1}{2}|V(H_j \setminus V(M_j))|,$$

then  $H_j \setminus V(M_j)$  has a Hamiltonian cycle. But



$$\begin{aligned}
\delta(H_j \setminus V(M_j)) &\geq \delta(G) - (j-1) - |V(M_j)| \\
&\geq \delta(G) - (r-2) - |V(M_j)| \\
&= \delta(G) - r + 2 - |V(M_j)|.
\end{aligned}$$

Also  $|V(H_j \setminus V(M_j))| = 2n - |V(M_j)|$ . Therefore

$$\begin{aligned}
&\delta(H_j \setminus V(M_j)) - \frac{1}{2}|V(H_j \setminus V(M_j))| \\
&\geq \delta - r + 2 - |V(M_j)| - n + \frac{1}{2}|V(M_j)| \\
&= \delta - r + 2 - n - \frac{1}{2}|V(M_j)| \\
&\geq \delta - r + 2 - n - \frac{1}{2}r \\
&= \delta - \frac{3r}{2} + 2 - n \\
&\geq 0, \text{ since } \delta \geq n + \frac{3r}{2} - 2.
\end{aligned}$$

Therefore  $H_j \setminus V(M_j)$  has a Hamilton cycle (which is necessarily of even length). Let  $F_j$  consist of  $M_j$  together with alternate edges of the Hamiltonian cycle. Since  $M_j$  was a maximal matching in  $G_r \setminus (M_1 \cup \dots \cup M_{j-1})$ , it follows that  $F_j$  contains no edge of  $M_{j+1} \cup \dots \cup M_r$ . This shows that a suitable  $F_j$  does exist.

The graph  $G \setminus (\bigcup_{i=1}^{r-1} F_i)$  has exactly  $r$  vertices of maximum degree, and each of these  $r$  vertices is joined to at most one other vertex of maximum degree. Therefore by Lemma 2.2,  $G \setminus (\bigcup_{i=1}^{r-1} F_i)$  is Class 1. Working back, it follows that  $G$  is also Class 1.

Proof of Theorem 8.1. Suppose  $G$  is critical but satisfies the inequality. Then, by Lemma 2.6,

$$\delta(G) \geq \Delta - r + 2,$$

from which it follows that the inequality of Theorem 8.6 holds.

Then  $G$  is Class 1, a contradiction. This proves Theorem 8.1.

### 8.3. Proof of Theorem 8.5.

It is convenient to prove Theorem 8.5 here, as it is used in the proof of Theorem 8.2 (and later in the proof of Theorem 9.2).

Lemma 8.1. Let  $G$  be a graph with  $|V(G)| = 2n+1, |E(G)| \leq n \Delta(G)$   
and let  $G$  have  $r$  vertices of maximum degree. If  $\Delta = \Delta(G) \geq 2n-r+1$ ,  
let  $t = \Delta - 2n + r$ . Let  $v$  be a vertex of degree  $\Delta$ . Then there exists  
a set  $X$  of vertices with  $v \notin X$  such that

$$d^*(v) \leq \left( \sum_{x \in X} \Delta - 1 - d(x) \right) + |\{x \in X: vx \notin E(G) \text{ and } d(x) \leq \Delta-1\}|$$

and

$$|X| \geq \begin{cases} d^*(v) + 1 - t & \text{if } \Delta \geq 2n-r+1, \\ d^*(v) + 1 & \text{if } \Delta \leq 2n-r. \end{cases}$$

Proof.  $\text{def}(G) = (2n+1) \Delta(G) - 2|E(G)| \geq (2n+1) \Delta - 2n\Delta = \Delta.$

There are  $r$  vertices of degree  $\Delta$ , so there are  $2n+1-r$  vertices of degree  $\leq \Delta-1$ . Let the excess deficiency  $\epsilon(G)$  be defined by

$$\epsilon(G) = \sum_{w \in V(G): d(w) \leq \Delta-1} (\Delta-1 - d(w)).$$

Then  $\epsilon(G) = \text{def}(G) - (2n+1-r) \geq \Delta - 2n + r - 1.$

Let  $v$  be a vertex of degree  $\Delta$ . Since  $d(v) = \Delta$  and  $v$  is joined to  $d^*(v)$  vertices of degree  $\Delta$ ,  $v$  is non-adjacent to  $r - d^*(v) - 1$  vertices of degree  $\Delta$ . But  $v$  is non-adjacent to  $2n-\Delta$  vertices altogether, and so is non-adjacent to  $(2n-\Delta) - (r-d^*(v)-1) = 2n-\Delta-r + d^*(v) + 1$  vertices of degree at most  $\Delta - 1$ .

Let

$$X = \{x \in V(G) : \text{either } d(x) < \Delta-1 \text{ or } d(x) = \Delta-1 \text{ and } xv \notin E(G)\}.$$

Then

$$\begin{aligned} & \sum_{x \in X} (\Delta-1-d(x)) + |\{x \in X : vx \notin E(G) \text{ and } d(x) \leq \Delta-1\}| \\ &= \varepsilon(G) + |\{x \in X : vx \notin E(G) \text{ and } d(x) \leq \Delta-1\}| \\ &\geq (\Delta-2n-1+r) + (2n-\Delta-r+d^*(v)+1) \\ &= d^*(v). \end{aligned}$$

Also

$$\begin{aligned} |X| &\geq |\{x : d(x) \leq \Delta-1 \text{ and } xv \notin E(G)\}| \\ &= 2n - \Delta - r + d^*(v) + 1, \text{ from above,} \\ &\geq \begin{cases} 2n - (2n-r+t) - r + d^*(v) + 1 & \text{if } \Delta \geq 2n-r+1, \\ d^*(v) + 1 & \text{if } \Delta \leq 2n-r, \end{cases} \\ &= \begin{cases} d^*(v) + 1 - t & \text{if } \Delta \geq 2n - r + 1, \\ d^*(v) + 1 & \text{if } \Delta \leq 2n-r. \end{cases} \end{aligned}$$

This proves Lemma 8.1.

Lemma 8.2. Let B be a bipartite graph. Let  $(x_1, \dots, x_q)$  and  $(w_1, \dots, w_q)$  be two sequences of vertices of B, where  $\{x_1, \dots, x_q\} \cap \{w_1, \dots, w_q\} = \emptyset$  and  $w_1, \dots, w_q$  are all distinct.  
Let m be the largest value of j for which there exist indices  $i_1, \dots, i_j$  with  $1 \leq i_1 < \dots < i_j \leq q$  and  $x_{i_1} = \dots = x_{i_j}$ . Let  $p \geq \max(q, m + \Delta(B) + 1)$ . Then we can partition the edge-set of B into matchings  $M_1, \dots, M_p$ , where, for some permutation  $\pi$  of  $(1, \dots, q)$ , no edge of  $M_i$  is incident with either  $x_i$  or  $w_{\pi(i)}$ .

Proof. We may suppose that  $q \geq 1$  (otherwise the lemma follows from Lemma 2.12.). We introduce two new vertices  $a$  and  $b$ , joining  $b$  to each of  $w_1, \dots, w_q$  by a single edge, and, for each  $x \in \{x_1, \dots, x_q\}$ , joining  $a$  to  $x$  by a number of edges equal to the number of times  $x$  appears in the sequence  $(x_1, \dots, x_q)$ , and, finally, joining  $a$  to  $b$  by  $p-q$  edges. Denote the graph thus formed by  $J$  ( $J$  may not be bipartite).

The graph  $J$  has two vertices,  $a, b$ , of maximum degree  $p$ , and the remaining vertices satisfy  $d_J(v) \leq m + \Delta(B) = p-1$ . All multiple edges are incident with the one vertex  $a$ , and, since  $q \geq 1$ , there is a vertex  $w_1$  joined to  $b$  but not to  $a$ . Since  $\{x_1, \dots, x_q\} \cap \{w_1, \dots, w_q\} = \emptyset$ ,  $J$  does not contain a subgraph on 3 vertices with  $p+1$  edges. Thus  $J$  satisfies Lemma 5.2 and so  $J$  is Class 1. Therefore we can colour  $J$  with  $p$  colours, say  $c_1, \dots, c_p$ .

Denote the colours used on the edges joining  $a$  to  $b$  by  $c_{q+1}, \dots, c_p$  and denote the colours on the edges joining  $a$  to each  $x \in \{x_1, \dots, x_q\}$  by  $c_{i_1}, c_{i_2}, \dots, c_{i_s}$ , where  $i_1, i_2, \dots, i_s$  are the indices  $i$  for which  $x_i = x$  ( $1 \leq i \leq q$ ); let  $\pi(i)$  be such that the edge  $bw_{\pi(i)}$  is coloured  $c_i$  ( $1 \leq i \leq q$ ). For  $1 \leq i \leq p$ , let  $M_i$  be the set of edges of  $B$  coloured  $c_i$ . Then  $M_1, \dots, M_p$  are the required matchings (clearly  $M_i$  contains no edge incident with  $x_i$  or  $w_{\pi(i)}$ ).

This proves Lemma 8.2.

Lemma 8.3. Let  $V_1, V_2, \dots, V_p$  be sets of vertices of a graph  $G$  and suppose that there are partial matchings  $M'_1, M'_2, \dots, M'_p$  such that

$$(i) \quad \bigcup_{i=1}^p M'_i = E(G),$$

and  $(ii) \quad M'_i$  contains no edge incident with a vertex of  $V_i$  ( $1 \leq i \leq p$ ).

Then there are partial matchings  $M_0, M_1, \dots, M_p$  such that

$$(i)' \quad \bigcup_{i=1}^p M_i = E(G),$$

$(ii)' \quad M_i$  contains no edge incident with a vertex of  $V_i$  ( $1 \leq i \leq p$ ),

and  $(iii)' \quad M_i$  is a partial matching which is maximal (by inclusion) in the graph  $(V(G), M_i \cup \dots \cup M_p)$ , subject to the proviso that  $(ii)'$  is satisfied, for  $1 \leq i \leq p$ .

Proof. Let  $M_1$  be a maximal matching in  $G$  containing  $M'_1$  but containing no edge incident with  $V_1$ . Proceeding inductively, let  $M_i$  be a maximal matching in  $(V(G), E(G) \setminus (M_1 \cup \dots \cup M_{i-1}))$  which contains  $M'_i \setminus (M_1 \cup \dots \cup M_{i-1})$  but contains no edge incident with  $V_i$ . Clearly we obtain  $M_1, \dots, M_p$  satisfying  $(i)'$ ,  $(ii)'$  and  $(iii)'$ .

We are now in a position to prove Theorem 8.5.

Proof of Theorem 8.5. The sufficiency follows from Lemma 2.7.

To prove the necessity assume that  $\delta(G)$  satisfies the inequality and that  $|E(G)| \leq n \Delta(G)$ . We shall show that  $G$  is Class 1.

The essential idea of the proof is to remove a set of 1-factors and near 1-factors from  $G$  in such a way that, in the resulting graph, each vertex of maximum degree has at most one other vertex of maximum degree adjacent to it. Then the necessity follows from a repeated application of Lemma 2.4.

Let  $v$  be a vertex with  $d(v) = \Delta(G)$ . Let  $q = d^*(v) - 1$ . Let  $X$  be a set of vertices such that

$$q = d^*(v) - 1 \leq \sum_{x \in X} (\Delta - 1 - d(x)) + |\{x \in X: vx \notin E(G) \text{ and } d(x) \leq \Delta - 1\}|$$

and

$$|X| = \begin{cases} d^*(v) + 1 - t & \text{if } \Delta \geq 2n - r + 2, \\ q & \text{if } \Delta \leq 2n - r + 1. \end{cases}$$

It follows easily from Lemma 8.1 that such a set  $X$  exists. Let

$(x_1, \dots, x_q)$  be a sequence of elements of  $X$  such that

$\{x_1, \dots, x_q\} = X$  and, if  $x \in X$ , then

$$|\{i: 1 \leq i \leq q \text{ and } x_i = x\}| \leq \Delta - 1 - d(x) + \begin{cases} 1 & \text{if } vx \notin E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let  $W$  be the set of vertices of degree  $\Delta$ . Let  $H$  denote the subgraph of  $G$  induced by  $(X \cup W) \setminus \{v\}$ .

Let  $M_0$  be a maximal (by inclusion) matching of  $H$ . Let  $L$  and  $R$  be sets of vertices of  $G$  such that

$$(X \cup W) \setminus \{v\} = L \cup R,$$

$$|L| \leq |R| \leq |L| + 1, \text{ and}$$

each edge of  $M_0$  joins a vertex of  $L$  to a vertex of  $R$ . Let  $B(L, R)$  be the bipartite subgraph of  $H$  induced by  $H$  with bipartition  $(L, R)$ .

$$\text{Let } \ell = \begin{cases} \left\lceil \frac{1}{2} (r + q + t - 1) \right\rceil & \text{if } \Delta \geq 2n - r + 2, \\ \left\lceil \frac{1}{2} (q + r + 1) \right\rceil & \text{if } \Delta \leq 2n - r + 1. \end{cases}$$

Let  $M_0, M_1, \dots, M_\ell$  be pairwise edge-disjoint partial matchings of  $B(L, R)$  such that

$$(Mi) \quad E(B(L,R)) = M_0 \cup \dots \cup M_\ell,$$

(Mii) for  $1 \leq i \leq q$ ,  $M_i$  contains no edge incident with either  $x_i$  or

$w_i$ , where  $\{w_1, \dots, w_q\}$  is a set of vertices of  $W$  joined to  $v$ .

and (Miii) for  $0 \leq i \leq \ell$ ,  $M_i$  is maximal (by inclusion) in the graph

$$(L \cup R, M_i \cup \dots \cup M_\ell), \text{ subject to (Mii).}$$

It follows from Lemma 8.2 that  $M_0, \dots, M_\ell$  exist satisfying (Mi) and (Mii), for the maximum degree in  $H$  is at most

$$\begin{cases} \left\lfloor \frac{1}{2}((r-1) + (q-t+2)) \right\rfloor & \text{if } \Delta \geq 2n-r+2, \\ \left\lfloor \frac{1}{2}((r-1) + q) \right\rfloor & \text{if } \Delta \leq 2n-r+1, \end{cases}$$

and the greatest value of  $j$  for which there exist indices  $i_1, \dots, i_j$  with  $1 \leq i_1 < \dots < i_j \leq q$  and  $x_{i_1} = \dots = x_{i_j}$  is

$$\begin{cases} q - (q-t-1) & \text{if } \Delta \geq 2n-r+2, \\ 1 & \text{if } \Delta \leq 2n-r+1. \end{cases}$$

and therefore the number  $p$  of that lemma is given by

$$\begin{aligned} 1 + (t-1) + \left\lfloor \frac{1}{2}(r + q - t + 1) \right\rfloor & \quad \text{if } \Delta \geq 2n-r+2, \\ 1 + 1 + \left\lfloor \frac{1}{2}(r + q - 1) \right\rfloor & \quad \text{if } \Delta \leq 2n-r+1. \end{aligned}$$

For our purposes, we take  $p = \ell + 1$ .

It follows from Lemma 8.3 that (Miii) can be satisfied also.

Our next step removes all edges of  $M_1, \dots, M_q$ , creates vertices  $x \in \{x_1, \dots, x_q\}$  of maximum degree and leaves  $v$  joined to only one vertex of maximum degree. We describe next how we carry this step out.

Let  $F_1, F_2, \dots, F_q$  be  $q$  edge-disjoint near 1-factors of  $G$  such that, for  $1 \leq i \leq q$ ,  $F_i$  contains  $M_i$  and  $w_i v$  but does not contain any edge incident with  $x_i$ , nor any edge of  $(M_0 \cup \dots \cup M_{i-1}) \cup (M_{i+1} \cup \dots \cup M_\ell)$ . To see that such near 1-factors exist, suppose that  $F_1, \dots, F_{i-1}$  have been chosen for some  $i$ ,  $1 \leq i \leq q$ . We show that  $F_i$  can be chosen. Consider the graph

$$J_i = (G \setminus (V(M_i) \cup \{v, w_i\})) \setminus (F_1 \cup \dots \cup F_{i-1} \cup M_0).$$

First observe that our assumptions imply that

$$|V(J_i)| = |V(G)| - |V(M_i)| - 2 = 2n-1 - |V(M_i)|, \text{ and}$$

$$\begin{aligned} \delta(J_i) &\geq \delta(G) - |V(M_i)| - 2 - (i-1) - 1 \\ &= \delta(G) - |V(M_i)| - i - 2. \end{aligned}$$

Therefore

$$\begin{aligned} &\delta(J_i) - \frac{1}{2} |V(J_i)| \\ &\geq \delta(G) - |V(M_i)| - i - 2 - n + \frac{1}{2} + \frac{1}{2} |V(M_i)| \\ &= \delta(G) - n - \frac{1}{2} |V(M_i)| - i - \frac{3}{2} \\ &\geq \begin{cases} \delta(G) - n - \frac{1}{2}(r + q - t + 1) - q - \frac{3}{2} & \text{if } \Delta \geq 2n-r+2, \\ \delta(G) - n - \frac{1}{2}(r + q - 1) - q - \frac{3}{2} & \text{if } \Delta \leq 2n-r+1, \end{cases} \\ &= \begin{cases} \delta - n - \frac{r}{2} - \frac{3q}{2} + \frac{t}{2} - 2 & \text{if } \Delta \geq 2n-r+2, \\ \delta - n - \frac{r}{2} - \frac{3q}{2} - 1 & \text{if } \Delta \leq 2n-r+1, \end{cases} \\ &\geq \begin{cases} \delta - n - 2r + \frac{t}{2} + 1 & \text{if } \Delta \geq 2n-r+2, \\ \delta - n - 2r + 2 & \text{if } \Delta \leq 2n-r+1, \end{cases} \\ &\quad \text{since } q = d^*(v) - 1 \leq r-2, \\ &\geq \begin{cases} \frac{r}{2} + \frac{t}{4} + \frac{1}{4} & \text{if } \Delta \geq 2n-r+2, \\ \frac{r}{2} + 1. & \text{if } \Delta \leq 2n-r+1, \end{cases} \\ &\geq 0. \end{aligned}$$



It follows from Lemma 7.5 that  $J_i$  has a Hamiltonian cycle, and therefore that  $(G \setminus (F_1 \cup \dots \cup F_{i-1}))$  has a near 1-factor  $F_i$  containing  $w_i v$ , containing  $M_i$ , but not containing any edge incident with  $x_i$ , nor any edge of  $M_0$ ; it follows from (Miii) and the fact that  $M_j \subset F_j$  ( $1 \leq j \leq i-1$ ) that  $F_i$  also contains no edge of  $(M_1 \cup \dots \cup M_{i-1}) \cup (M_{i+1} \cup \dots \cup M_\ell)$ .

The graph  $G \setminus (F_1 \cup \dots \cup F_q)$  has at most

$$\begin{cases} r + d^*(v) + 1 - t & \text{if } \Delta \geq 2n - r + 2, \\ r + q & \text{if } \Delta \leq 2n - r + 1, \end{cases}$$

vertices of maximum degree, but  $v$  is adjacent to only one of them.

Therefore by Lemma 2.4,  $(G \setminus v) \setminus (F_1 \cup \dots \cup F_q)$  and  $G \setminus (F_1 \cup \dots \cup F_q)$  have the same Class. Let  $S = (G \setminus v) \setminus (F_1 \cup \dots \cup F_q)$ . We need to show that  $S$  is Class 1. Note that  $|V(S)| = 2n$ , so is even, and that  $\delta(S) \geq \delta(G) - 1 - q$ .

We now remove all the remaining edges in  $B(L, R)$  except for those in the maximal partial matching  $M_0$ . We describe now how we carry this step out.

Let  $F_{q+1}, \dots, F_\ell$  be  $\ell - q$  edge disjoint 1-factors of  $S$  such that, for  $q + 1 \leq i \leq \ell$ ,  $F_i$  contains  $M_i$  but does not contain any edge of  $M_0$ . From (Miii) and the fact that  $M_j \subset F_j$  ( $1 \leq j \leq i-1$ ), it follows that  $F_i$  will also not contain any edge of  $(M_1 \cup \dots \cup M_{i-1}) \cup (M_{i+1} \cup \dots \cup M_\ell)$  either. To see that such 1-factors exist, suppose that  $F_{q+1}, \dots, F_{i-1}$  have been chosen for some  $i$ ,  $q+1 \leq i \leq \ell$ . We show that  $F_i$  can be chosen.

Consider the graph

$$J_i' = (S \setminus V(M_i)) \setminus (F_{q+1} \cup \dots \cup F_{i-1} \cup M_0).$$

Then we have

$$\begin{aligned}
\delta(J_i') &\geq \delta(G) - 1 - q - |V(M_i)| - (i-1-q) - 1 \\
&= \delta(G) - 1 - |V(M_i)| - i \\
&\geq \delta(G) - 1 - |V(M_i)| - \ell.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\delta(J_i') - \frac{1}{2} |V(J_i')| \\
&\geq \delta(G) - |V(M_i)| - \ell - 1 - \frac{1}{2}(2n - |V(M_i)|) \\
&= \delta(G) - \frac{1}{2}|V(M_i)| - \ell - n - 1 \\
&\geq \begin{cases} \delta - \frac{1}{2}(r+q-t+1) - \left\lceil \frac{1}{2}(q+r+t-1) - n+1 \right\rceil & \text{if } \Delta \geq 2n-r+2, \\ \delta - \frac{1}{2}(r+q-1) - \left\lceil \frac{1}{2}(q+r+1) \right\rceil - n+1 & \text{if } \Delta \leq 2n-r+1, \end{cases} \\
&\geq \delta - r - q - n - \frac{3}{2} \\
&\geq \delta - r - (r-2) - n - \frac{3}{2}, \text{ since } q = d^*(v) - 1 \leq r-2, \\
&\geq \begin{cases} (n + \frac{5}{2}r - \frac{1}{4}t - \frac{3}{4}) - 2r - n + \frac{1}{2} & \text{if } \Delta \geq 2n-r+2, \\ (n + \frac{5}{2}r - 1) - 2r - n + \frac{1}{2} & \text{if } \Delta \leq 2n-r+1, \end{cases} \\
&= \begin{cases} \frac{1}{2}r - \frac{1}{4}t - \frac{1}{4} & \text{if } \Delta \geq 2n-r+2, \\ \frac{1}{2}r - \frac{1}{2} & \text{if } \Delta \leq 2n-r+1, \end{cases} \\
&\geq 0.
\end{aligned}$$

It follows from Lemma 7.5 that  $J_i'$  has a Hamiltonian cycle and therefore that  $S \setminus (F_{q+1} \cup \dots \cup F_{i-1})$  has a 1-factor  $F_i$  containing  $M_i$ , but not containing any edge of  $(M_0 \cup \dots \cup M_{i-1}) \cup (M_{i+1} \cup \dots \cup M_\ell)$ .

Our next step removes all edges of  $L$ ; it follows that each vertex of  $L$  is then joined to at most one other vertex of  $B(L, R)$ . We describe now how we carry this step out.

Now let  $S^* = S \setminus (F_{q+1} \cup \dots \cup F_\ell)$ . Then  $\delta(S^*) \geq \delta(G) - 1 - q - (\ell - q) = \delta(G) - \ell - 1$ . Consider the subgraph  $S_L^*$  of  $S^*$  induced by  $L$ . Let  $s = |L|$ .

$$\begin{aligned}
\text{Then } s &= |L| \\
&= \left\lfloor \frac{1}{2} |X \cup W| \setminus \{v\} \right\rfloor \\
&= \begin{cases} \left\lfloor \frac{1}{2} \{(r-1) + (q-t+2)\} \right\rfloor & \text{if } \Delta \geq 2n-r+2, \\ \left\lfloor \frac{1}{2} \{(r-1)+q\} \right\rfloor & \text{if } \Delta \leq 2n-r+1, \end{cases} \\
&= \begin{cases} \left\lfloor \frac{1}{2}(r+q-t+1) \right\rfloor & \text{if } \Delta \geq 2n-r+2, \\ \left\lfloor \frac{1}{2}(r+q-1) \right\rfloor & \text{if } \Delta \leq 2n-r+1. \end{cases}
\end{aligned}$$

Let  $M_1^* \cup \dots \cup M_s^*$  be pairwise edge-disjoint partial matchings of  $S_L^*$  such that

$$(M_i^*) \quad E(S_L^*) = M_1^* \cup \dots \cup M_s^*,$$

(M<sub>i</sub><sup>\*</sup>) for  $1 \leq i \leq s$ ,  $M_i^*$  is maximal (by inclusion) in the graph  $(L, M_1^* \cup \dots \cup M_s^*)$ .

Let  $F_1^*, \dots, F_s^*$  be edge disjoint 1-factors of  $S^*/M_0$  such that

$F_i^*$  contains  $M_i^*$ ; (M<sub>i</sub><sup>\*</sup>) and the fact that  $M_j^* \subseteq F_j^*$  ( $1 \leq j \leq i-1$ ) ensures that  $F_i^*$  contains no edge of  $(M_1^* \cup \dots \cup M_{i-1}^*) \cup (M_{i+1}^* \cup \dots \cup M_s^*)$ ; since  $S^*/M_0$  contains no edge of  $M_0$ , none of  $F_1^*, \dots, F_s^*$  contain any edges of  $M_0$ .

To see that such 1-factors exist, for  $1 \leq i \leq s$ , suppose that  $F_1^*, \dots, F_{i-1}^*$  have been chosen. Consider the graph

$$J_i^* = (S^* \setminus V(M_i^*)) \setminus (F_1^* \cup \dots \cup F_{i-1}^* \cup M_0).$$

Then

$$\begin{aligned}
\delta(J_i^*) &\geq \delta(S^*) - |V(M_i^*)| - (i-1) - 1 \\
&\geq \delta(G) - \ell - 1 - |V(M_i^*)| - i \\
&\geq \delta(G) - \ell - 1 - |V(M_i^*)| - s.
\end{aligned}$$

Also

$$|V(J_i^*)| = 2n - |V(M_i^*)|.$$

Therefore

$$\delta(J_i^*) - \frac{1}{2}|V(J_i^*)|$$

$$\begin{aligned}
&\geq \delta(G) - \ell - 1 - s - |V(M_i^*)| - n + \frac{1}{2} |V(M_i^*)| \\
&= \delta(G) - n - \frac{1}{2} |V(M_i^*)| - \ell - s - 1 \\
&\geq \delta(G) - n - \frac{1}{2}s - \ell - s - 1 \\
&= \delta - n - \ell - \frac{3}{2}s - 1 \\
&\geq \begin{cases} \delta - n - \left\lfloor \frac{1}{2}(q+r+t-1) \right\rfloor - \frac{3}{2} \left\lfloor \frac{1}{2}(r+q-t+1) \right\rfloor - 1 & \text{if } \Delta \geq 2n-r+2, \\ \delta - n - \left\lfloor \frac{1}{2}(q+r+1) \right\rfloor - \frac{3}{2} \left\lfloor \frac{1}{2}(r+q-1) \right\rfloor - 1 & \text{if } \Delta \leq 2n-r+1, \end{cases} \\
&\geq \begin{cases} \delta - \frac{5}{4}r - n - \frac{5}{4}q + \frac{1}{4}t + \frac{7}{4} & \text{if } \Delta \geq 2n-r+2, \\ \delta - \frac{5}{4}r - n - \frac{5}{4}q + \frac{5}{4} & \text{if } \Delta \leq 2n-r+1, \end{cases} \\
&\geq \begin{cases} \delta - \frac{5}{4}r - n - \frac{5}{4}(r-2) + \frac{1}{4}t + \frac{7}{4} & \text{if } \Delta \geq 2n-r+2, \\ \delta - \frac{5}{4}r - n - \frac{5}{4}(r-2) + \frac{5}{4} & \text{if } \Delta \leq 2n-r+1, \end{cases} \\
&\geq \begin{cases} \delta - \frac{5}{2}r - n + \frac{1}{4}t + \frac{3}{4} & \text{if } \Delta \geq 2n-r+2, \\ \delta - \frac{5}{2}r - n + \frac{5}{4} & \text{if } \Delta \leq 2n-r+1, \end{cases} \\
&\geq 0.
\end{aligned}$$

It follows from Lemma 7.5 that  $J^*$  has a Hamiltonian cycle and therefore that  $S^* \setminus (F_1^* \cup \dots \cup F_{i-1}^*)$  has a 1-factor  $F_i^*$  containing  $M_i^*$ , but not containing any edge of  $M_0$  nor of  $(M_1^* \cup \dots \cup M_{i-1}^*) \cup (M_{i+1}^* \cup \dots \cup M_s^*)$ .

The graph  $H^* = S^* \setminus (F_1^* \cup \dots \cup F_s^*)$  has the same set  $L \cup R$  of vertices of maximum degree as had  $S$ . In  $H^*$  the vertices of  $L$  are joined to at most one vertex of maximum degree. The vertices of  $R$  which are not joined by an edge of  $M_0$  are pairwise non-adjacent in  $H^*$ , since  $M_0$  was chosen to be a maximal partial matching of  $H$ . Therefore by Lemma 2.4, the graph  $S^* \setminus (F_1^* \cup \dots \cup F_s^*)$  is Class 1. Working back it follows that  $G \setminus w$  is Class 1, as required. This proves Theorem 8.5.

#### 8.4. Proof of Theorem 8.2.

Theorem 8.5 is in itself the most significant step in the proof of Theorem 8.2; the following lemma follows easily from Theorem 8.5.

Lemma 8.4. Let  $G$  have  $2n+1$  vertices, of which  $r$  have maximum degree  $\Delta$ . Let

$$\Delta \geq \begin{cases} n + \frac{7}{2}r - \frac{1}{4}t - \frac{11}{4} & \text{if } \Delta = 2n+1-r+t \text{ and } t > 0, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r, \end{cases}$$

If (i)  $G$  is critical,

then (ii)  $|E(G)| = n\Delta + 1$ .

Proof. Suppose  $G$  is critical and satisfies the inequality.

Then, by Lemma 2.6,

$$\delta(G) \geq \Delta - r + 2,$$

from which it follows that the inequality of Theorem 8.5 holds.

Therefore  $|E(G)| > n\Delta(G)$ . But since  $G$  is critical, it follows from

Lemma 2.7 that  $|E(G)| = n\Delta(G) + 1$ .

This proves Lemma 8.4.

For positive integers  $r$  and  $n$ , let  $f(n,r)$  be defined by

$$f(n,r) = \begin{cases} \frac{6}{5}n + \frac{13}{5}r - 2 & \text{if } r \geq \frac{2}{9}n + \frac{10}{9}, \\ n + \frac{7}{2}r - 3 & \text{if } r \leq \frac{2}{9}n + \frac{10}{9}. \end{cases}$$

There is no ambiguity in this definition, for if  $r = \frac{2}{9}n + \frac{10}{9}$  then

$$\frac{6}{5}n + \frac{13}{5}r - 2 = n + \frac{7}{2}r - 3.$$

Lemma 8.5.

$$(i) \quad f(n,r) = \min \left( \frac{6}{5}n + \frac{13}{5}r - 2, n + \frac{7}{2}r - 3 \right),$$

(ii) The inequality in Theorem 8.2 and in Lemma 8.4 may be put in the form

$$\Delta \geq f(n,r).$$

(iii)  $f(n,r)$  is an increasing function of  $n$  and of  $r$ .

Proof. It is easy to verify that

$$r \geq \frac{2}{9}n + \frac{10}{9} \quad \text{as} \quad \frac{6}{5}n + \frac{13}{5}r - 2 \geq 2n+2-r,$$

that

$$r \geq \frac{2}{9}n + \frac{10}{9} \quad \text{as} \quad n + \frac{7}{2}r - 3 \geq 2n+2-r,$$

and that

$$r \geq \frac{2}{9}n + \frac{10}{9} \quad \text{as} \quad n + \frac{7}{2}r - 3 \geq \frac{6}{5}n + \frac{13}{5}r - 2,$$

or, in other words, that

$$r \geq \frac{2}{9}n + \frac{10}{9} \quad \text{as} \quad n + \frac{7}{2}r - 3 \geq \frac{6}{5}n + \frac{13}{5}r - 2 \geq 2n+2-r.$$

(i) now follows immediately.

It also follows immediately that the condition

$$"\Delta \geq \frac{6}{5}n + \frac{13}{5}r - 2 \quad \text{if} \quad \Delta \geq 2n+2-r"$$

can be rewritten

$$"\Delta \geq f(n,r) \quad \text{if} \quad \Delta \geq 2n+2-r"$$

and that the condition

$$"\Delta \geq n + \frac{7}{2}r - 3 \quad \text{if} \quad \Delta \leq 2n+2-r"$$

can be rewritten

$$"\Delta \geq f(n,r) \quad \text{if } \Delta \leq 2n+2-r."$$

Consequently the inequality

$$"\Delta \geq \begin{cases} \frac{6}{5}n + \frac{13}{5}r - 2 & \text{if } \Delta \geq 2n+2-r, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r, \end{cases} "$$

can be rewritten

$$\Delta \geq f(n,r).$$

But as indicated after the statement of Theorem 8.2, the combined inequality above is equivalent to the inequality of Theorem 8.2. This proves (ii).

(iii) follows immediately from the definition of  $f(n,r)$ .

This proves Lemma 8.5.

Lemma 8.6.      Let  $G$  have  $2n+1$  vertices,  $r$  of them having maximum degree  $\Delta$ . Then the following are equivalent.

$$(ii) \quad |E(G)| = n\Delta + 1,$$

$$(iv) \quad \text{def}(G) = \Delta - 2.$$

$$\begin{aligned} \text{Proof.} \quad \text{def}(G) &= \Delta|V(G)| - 2|E(G)| \\ &= \Delta(2n+1) - 2|E(G)| \\ &= \Delta - 2(|E(G)| - n\Delta). \end{aligned}$$

Therefore if  $|E(G)| = n\Delta + 1$ , then  $\text{def}(G) = \Delta - 2$ .

Conversely if  $\text{def}(G) = \Delta - 2$ , then  $|E(G)| = n\Delta + 1$ .

We now prove the converse of Lemma 8.4.

Lemma 8.7.      Let G have  $2n+1$  vertices, of which  $r$  have maximum degree  $\Delta$ . Let

$$\Delta \geq \begin{cases} n + \frac{7}{2}r - \frac{1}{4}t - \frac{11}{4} & \text{if } \Delta = 2n+1-r+t \text{ and } t > 0, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r. \end{cases}$$

If (ii)  $|E(G)| = n\Delta+1$ ,  
then (i) G is critical.

Proof. Since  $|E(G)| = n\Delta+1 > \left\lfloor \frac{|V(G)|}{2} \right\rfloor \Delta$ , by Lemma 2.7, G is Class 2.

Suppose G is not critical. Then G contains a critical subgraph  $G^*$  of the same maximum degree  $\Delta$  with  $r^*$  ( $\leq r$ ) vertices of maximum degree.

By Theorem 8.1, since

$$n + \frac{5}{2}r - 4 \leq \begin{cases} n + \frac{7}{2}r - \frac{1}{4}t - \frac{11}{4} & \text{if } \Delta = 2n+1-r+t \text{ and } t > 0, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r, \end{cases}$$

it follows that  $|V(G^*)|$  is not even. Let  $|V(G^*)| = 2n^*+1$  for some  $n^* \leq n$ . By Lemma 8.5 (iii) (in the notation of that lemma),

$$\Delta(G^*) = \Delta(G) \geq f(n,r) \geq f(n^*,r^*).$$

Therefore by Lemma 8.5 (ii) and Lemma 8.4,  $|E(G^*)| = n^*\Delta+1$ .

By Lemma 8.6, the deficiencies of both G and  $G^*$  are  $\Delta-2$ , so the number of edges that can be added to  $G^*$  in forming G is at most

$$\binom{(2n+1)-(2n^*+1)}{2} = (n-n^*)(n-n^*-1).$$

However

$$|E(G)| - |E(G^*)| = \Delta(n-n^*),$$

so it follows that



$$\Delta(n-n^*) \leq (n-n^*)(n-n^*-1),$$

and so, if  $n \neq n^*$ , then

$$\Delta \leq n-n^*-1 < n.$$

However the inequalities of the lemma imply that  $\Delta \geq n+2$ . This is a contradiction. Therefore  $n = n^*$ , and so  $|E(G^*)| = n\Delta+1 = |E(G)|$  and  $|V(G^*)| = 2n+1 = |V(G)|$ . Therefore  $G = G^*$  and so  $G$  is critical.

This proves Lemma 8.7.

Combining Lemmas 8.4 and 8.7 we have

Lemma 8.8.      Let  $G$  have  $2n+1$  vertices of which  $r$  have maximum degree  $\Delta$ . Let

$$\Delta \geq \begin{cases} n + \frac{7}{2}r - \frac{1}{4}t - \frac{11}{4} & \text{if } \Delta = 2n+1-r+t \text{ and } t > 0, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r. \end{cases}$$

Then the following are equivalent:

- (i)  $G$  is critical,
- (ii)  $|E(G)| = n\Delta+1.$

The next two lemmas show that (i) and (iii) in Theorem 8.2 are equivalent.

Lemma 8.9.      Let  $G$  have  $2n+1$  vertices  $r$  having maximum degree  $\Delta$ .

Let  $\Delta \geq n+r-2$ .

If

- (i)  $G$  is critical,

then

(iii) G is (r-2) edge-connected and Class 2, and

$$|E(G)| \leq n\Delta + 1.$$

Proof. Clearly G is Class 2 and, from Lemma 2.7,  $|E(G)| \leq n\Delta + 1$ .

Let S be a set of vertices of G with  $|S| \leq n$ . By Lemma 2.6,  $\delta(G) \geq \Delta - r + 2 \geq n$ . Therefore the number of edges between S and  $V(G) - S$  is at least

$$|S| (\delta - |S| + 1)$$

$$\geq \min(\delta, n(\delta - n + 1))$$

$$\geq n$$

$$\geq r - 2.$$

Therefore  $\lambda(G) \geq r - 2$  as required.

Lemma 8.10.

Let G have  $2n+1$  vertices, r having maximum degree

$\Delta$ . Let

$$\Delta \geq \begin{cases} n + \frac{7}{2}r - \frac{1}{4}t - \frac{11}{4} & \text{if } \Delta = 2n+1-r+t \text{ and } t > 0, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r. \end{cases}$$

If

(iii) G is (r-2)-edge-connected and Class 2 and  $|E(G)| \leq n\Delta + 1$ ,

then

(i) G is critical.

Proof. Suppose G satisfies (iii). Let  $G^*$  be a critical subgraph of G with the same maximum degree  $\Delta$ . Since

$$n + \frac{5}{2}r - 4 \leq \begin{cases} n + \frac{7}{2}r - \frac{1}{4}t - \frac{11}{4} & \text{if } \Delta = 2n+1-r+t \text{ and } t > 0, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r, \end{cases}$$

it follows from Theorem 8.1 that  $|V(G^*)|$  is odd. Let  $|V(G^*)| = 2n^*+1$ . Let  $G^*$  have  $r^*$  ( $\leq r$ ) vertices of maximum degree. By Lemma 8.5 (iii) (in the notation of that lemma),

$$\Delta(G^*) = \Delta(G) \geq f(n, r) \geq f(n^*, r^*).$$

Therefore, by Lemma 8.5 (ii) and Lemma 8.4,  $|E(G^*)| = n^* \Delta + 1$ .

As remarked in the proof of Lemma 8.1, the excess deficiency  $\varepsilon(G^*)$ , satisfies

$$\begin{aligned} \varepsilon(G^*) &= \sum_{(v: d_{G^*}(v) < \Delta)} (\Delta - 1 - d_{G^*}(v)) \\ &= \text{def}(G^*) - (2n^*+1-r^*) \\ &= (\Delta-2) - (2n^*+1-r^*), \text{ by Lemma 8.6,} \\ &= \Delta - 2n^* + r^* - 3 \\ &\leq r^* - 3, \end{aligned}$$

since  $\Delta \leq 2n^*$ , as  $|V(G^*)| = 2n^*+1$  and  $\Delta = \Delta(G^*)$ . If  $n^* < n$  then the number of edges of  $G$  joining  $V(G^*)$  to  $V(G) \setminus V(G^*)$  is at most

$$(r^*-3) + (r-r^*) = r-3,$$

for otherwise  $G$  would have more than  $r$  vertices of maximum degree.

However this contradicts the hypothesis that  $G$  is  $(r-2)$ -edge-connected.

Therefore  $n = n^*$ , so  $|E(G^*)| = n\Delta + 1$ .

Since  $G^*$  is a subgraph of  $G$ , and since  $|E(G)| \leq n\Delta + 1 = |E(G^*)|$ , it follows that  $G = G^*$ , and so  $G$  is critical, as required.

This proves Lemma 8.10.

Lemma 8.11.    Let  $G$  have  $2n+1$  vertices,  $r(\leq n)$  of them having maximum degree  $\Delta$ .

If

$$(iv) \text{ def } (G) = \Delta - 2,$$

then the edge-connectivity  $\lambda(G)$  satisfies

$$(v) \quad \lambda(G) \geq 2n+2-r.$$

Proof. As remarked in the proof of Lemma 8.1, the excess deficiency  $\epsilon(G)$  of  $G$  satisfies

$$\epsilon(G) = \text{def}(G) - (2n+1-r).$$

Therefore

$$\epsilon(G) = (\Delta - 2) - (2n+1-r) = \Delta - 2n + r - 3.$$

Therefore

$$\delta(G) \geq (\Delta - 1) - (\Delta - 2n + r - 3) = 2n + 2 - r.$$

Let  $S$  be a set of vertices of  $G$  with  $|S| \leq n$ . Since  $\delta(G) \geq n$ , the number of edges between  $S$  and  $V(G) - S$  is at least

$$\begin{aligned} & |S| (\delta - |S| + 1) \\ & \geq \min (\delta, n(\delta - n + 1)) \\ & \geq \min (2n + 2 - r, n(n + 3 - r)) \\ & = 2n + 2 - r. \end{aligned}$$

Therefore  $\lambda \geq 2n + 2 - r$ , as required.

This proves Lemma 8.11.

Proof of Theorem 8.2. By Lemma 8.8, (i) and (ii) are equivalent if  $\Delta$  satisfies the inequalities of the theorem. By Lemma 8.6, (ii) and (iv) are also equivalent then, and, by Lemmas 8.9 and 8.10, (iii) is also equivalent to (i) then. By Lemma 8.11, each of these implies (v) then.

This proves Theorem 8.2.

### 8.5. Proofs of Theorems 8.3 and 8.4.

Proof of Theorem 8.3. Suppose  $G$  satisfies the hypotheses of the theorem. If  $G$  is Class 2, then  $G$  has a critical subgraph  $G^*$  with the same maximum degree  $\Delta$  and with  $r^*$  ( $\leq r$ ) vertices of maximum degree. Since

$$n + \frac{5}{2}r - 4 \leq \begin{cases} n + \frac{7}{2}r - \frac{1}{4}t - \frac{11}{4} & \text{if } \Delta = 2n+1-r+t \text{ and } t > 0, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r, \end{cases}$$

it follows from Theorem 8.1 that  $|V(G^*)|$  is not even. Let  $|V(G^*)| = 2n^*+1$ . By Lemma 8.5,

$$\Delta(G^*) = \Delta(G) \geq f(n, r) \geq f(n^*, r^*).$$

Therefore, by Theorem 8.2,  $|E(G^*)| = n^*\Delta + 1$ .

The excess deficiency  $\varepsilon(G^*)$  satisfies

$$\begin{aligned} \varepsilon(G^*) &= \text{def}(G^*) - (2n^*+1-r^*) \\ &= (\Delta-2) - (2n^*+1-r^*), \text{ by Theorem 8.2,} \\ &= \Delta - 2n^* + r^* - 3 \\ &\leq r^* - 3, \end{aligned}$$

since  $\Delta \leq 2n^*$ , as  $|V(G^*)| = 2n^*+1$  and  $\Delta = \Delta(G^*)$ . The number of edges of  $G$  joining  $V(G^*)$  to  $V(G) \setminus V(G^*)$  is at most

$$(r^*-3) + (r-r^*) = r-3,$$

for otherwise  $G$  would have more than  $r$  vertices of maximum degree.

However this contradicts the hypothesis that  $G$  is  $(r-2)$ -edge-connected.

Therefore  $G$  is Class 1, as required.

This proves Theorem 8.3.

#### Proof of Theorem 8.4.

Sufficiency. This follows from Lemma 2.7.

Necessity. Suppose  $G$  satisfies the hypotheses of the theorem and is Class 2. Then  $G$  has a critical subgraph  $G^*$  with the same maximum degree  $\Delta$  and with  $r^*(\leq r)$  vertices of maximum degree. By the same argument as in the proof of Theorem 8.3,  $|V(G^*)|$  is odd. Let  $|V(G^*)| = 2n^*+1$ , where  $n^* \leq n$ .

If  $n^* < n$  then we obtain a contradiction in the same way as in the proof of Theorem 8.3. Therefore  $n^* = n$ . By Theorem 8.2,  $|E(G^*)| = n\Delta+1$ , and so  $|E(G)| > n\Delta$ , as required.

This proves Theorem 8.4.

### 8.6. Proof of Theorem 8.7.

Sufficiency. By Lemma 2.7,  $G_1$  is Class 2 and so it follows that  $G$  is Class 2.

Necessity. Suppose  $G$  is Class 2. Then  $G$  has a  $\Delta$ -critical subgraph  $G^*$  with  $r^*(<r)$  vertices of maximum degree. By Theorem 8.1, since

$$n + \frac{5}{2}r - 4 \leq \begin{cases} n + \frac{7}{2}r - \frac{1}{4}t - \frac{11}{4} & \text{if } \Delta = 2n+1-r+t \text{ and } t>0, \\ n + \frac{7}{2}r - 3 & \text{if } \Delta \leq 2n+2-r \end{cases}$$

it follows that  $|V(G^*)|$  is not even. Let  $|V(G^*)| = 2n^* + 1$  for some  $n^* \leq n$ . By Lemma 8.5 (iii),

$$\Delta(G^*) = \Delta(G) \geq f(n, r) \geq f(n^*, r^*).$$

Therefore, by Lemma 8.5 (ii) and Lemma 8.4,  $|E(G^*)| = n^*\Delta + 1$ .

The excess deficiency of  $G^*$  is  $\Delta - 2 - (2n^* + 1 - r^*)$ , so the number of further edges which can be incident with vertices of  $G^*$  without forming more than  $r$  vertices of maximum degree is at most

$$\begin{aligned} & \Delta - 2 - (2n^* + 1 - r^*) + (r - r^*) \\ &= \Delta - 3 - 2n^* + r \\ &= (r - 3) + (\Delta^* - 2n^*) \\ &\leq r - 3. \end{aligned}$$

Clearly  $|V(G^*)| > |V(G \setminus G^*)|$ . Therefore the theorem is satisfied with  $V(G_1) = V(G^*)$  and  $G^*$  being a subgraph of  $G_1$ .

This proves Theorem 8.7.

## 9. Regular graphs of high degree are 1-factorizable

### 9.1 Introduction

It is a well known conjecture that if a regular graph  $G$  of order  $2n$  has degree  $d(G)$  satisfying  $d(G) \geq n$ , then  $G$  is the union of edge disjoint 1-factors. This conjecture is known to be true for  $d(G) = 2n - 1$  or  $2n - 2$  and we show here that it is also true for  $d(G) = 2n - 3$ ,  $2n - 4$  or  $2n - 5$  and for  $d(G) \geq \frac{6}{7} |V(G)|$ .

Conjecture 9.1 A regular graph of order  $2n$  and degree  $d(G)$  satisfying

$$d(G) \geq 2 \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \quad \text{is Class 1}.$$

The lower bound  $2 \left\lfloor \frac{n+1}{2} \right\rfloor - 1$  is best possible. A connected regular graph of order  $2n$  and degree  $2 \left\lfloor \frac{n+1}{2} \right\rfloor - 2$  which is of Class 2 can be formed for  $n = 2m + 1$ ,  $m \geq 2$ , from two copies of  $K_{2m+1}$  by removing one edge (say  $a_1b_1$  and  $a_2b_2$ ) from each and joining the two copies by edges  $a_1a_2$  and  $b_1b_2$ . The Petersen graph is an example of a connected regular Class 2 graph of order  $2n$  and degree  $2 \left\lfloor \frac{n+1}{2} \right\rfloor - 3$ .

It is well known that  $K_{2n}$  is Class 1, and a trivial consequence is that a regular graph of order  $2n$  and degree  $2n - 2$  is Class 1 (as any such graph can be formed by removing a 1-factor from  $K_{2n}$ ). Rosa and Wallis [R1] recently proved the case when  $d(G) = 2n - 4$  under the special circumstance that  $\bar{G}$  is Class 1. Häggkvist has showed us a sketch of a proof of the conjecture when  $d(G) \geq \frac{127}{128} |V(G)|$ . He also has proved that, given  $\epsilon > 0$ , there exists  $n$  such that if  $|V(G)| \geq n$  and even, and  $G$  is regular with  $d(G) \geq (\frac{1}{2} + \epsilon) |V(G)|$ , then  $G$  is 1-factorizable. Our



method and his bear no resemblance to each other.

In this chapter we have two main results. Theorems 9.1 and 9.2, which are both special cases of the conjecture.

Theorem 9.1 Let  $G$  be a regular graph of order  $2n$  and degree  
 $d(G) = 2n - 3, 2n - 4$  or  $2n - 5$ . Let  $d(G) \geq 2 \left\lfloor \frac{n+1}{2} \right\rfloor - 1$ . Then  $G$  is  
Class 1.

Theorem 9.2 Let  $G$  be a regular graph of order  $2n$  whose degree  
 $d(G)$  satisfies

$$d(G) \geq \frac{6}{7} |V(G)|.$$

Then  $G$  is Class 1.

Theorem 9.2 has an application on the subject of 'Intricacy' about which an interesting paper has recently been written by W. E. Opencomb [01]. Briefly, suppose we have a set of edge-disjoint 1-factors of  $K_{2n}$ . It may well be that this set of 1-factors cannot be completed to give a 1-factorization of  $K_{2n}$ . In that case, for some integer  $j = j(n)$ , it is certainly possible to partition the given set of edge-disjoint 1-factors into  $j$  parts in such a way that the set of 1-factors in each part of the partition can be extended to a 1-factorization of  $K_{2n}$ . The intricacy of this problem is the least  $j$  for which there always exists a partition into  $j$  parts, each of which can be extended to a 1-factorization of  $K_{2n}$ . The conjecture would imply that the intricacy of this problem was 2. Theorem 9.2 implies that it is no more than 7. In the notation of [01], we have:

Corollary 9.1. For  $n \geq 3$ ,

$$2 \leq \kappa(\text{Pack}^2 \xi(K_{2n})) \leq 7.$$

The upper bound, 7, replaces the upper bound  $\lceil \frac{2n-3}{3} \rceil$  given in [01].

Combining Theorem 9.1 and Theorem 9.2 with recent results of Faudree and Sheehan [F1] we obtain the following corollaries.

Corollary 9.2. Let  $2 \leq k \leq 4$ ,  $n \geq k$ . If  $G$  is a connected regular graph of degree  $k$  and order  $2n$ , and  $G \neq K_{3,3}$  if  $k = 3$ , then  $\bar{G}$  has a 1-factor  $F$  such that both  $G \cup F$  and  $\bar{G} \setminus F$  are Class 1.

Corollary 9.3. Let  $k \geq 2$ ,  $2n \geq \max(2(k^2 - k + 1), 7k + 14)$ . If  $G$  is a connected regular graph of degree  $k$  and order  $2n$ , then  $\bar{G}$  has a 1-factor  $F$  such that both  $G \cup F$  and  $\bar{G} \setminus F$  are Class 1.

Combining a slight extension of Conjecture 1 with a slightly generalized form of a conjecture of Faudree and Sheehan [F1], we have:

Conjecture 9.2. Let  $n \geq k \geq 2$ . If  $G$  is a regular graph of degree  $k$  and order  $2n$ , and  $G \neq K_{k,k}$  if  $k$  is odd, then  $\bar{G}$  has a 1-factor  $F$  such that both  $G \cup F$  and  $\bar{G} \setminus F$  are Class 1.

## 9.2 Preliminary results

We give here some lemmas used in the following sections.

Lemma 9.1 Let  $n \geq 1$ . Let  $G$  be a regular graph of order  $2n$ ,  $G \neq K_{2n}$ .  
Let  $w \in V(G)$ . Then  $G$  is Class 1 if and only if  $G \setminus w$  is Class 1.

### Proof

Necessity. If  $G$  is Class 1, then  $G$  can be edge-coloured with  $\Delta(G)$  colours. Therefore  $G \setminus w$  can be edge-coloured with  $\Delta(G)$  colours. Since  $G \neq K_{2n}$ , there is a vertex in  $G$  non-adjacent to  $w$ , so  $\Delta(G \setminus w) = \Delta(G)$ . Thus  $G \setminus w$  can be edge-coloured with  $\Delta(G \setminus w)$  colours, so  $G \setminus w$  is Class 1.

Sufficiency. If  $G \setminus w$  is Class 1, let  $G \setminus w$  be coloured with  $\Delta(G \setminus w)$  colours. As above,  $\Delta(G \setminus w) = \Delta(G)$ . The graph  $G \setminus w$  has  $\Delta(G)$  vertices of degree  $\Delta(G) - 1$  and  $|V(G \setminus w)|$  is odd. Therefore each colour is missing from exactly one vertex and each vertex of degree  $\Delta(G) - 1$  has exactly one colour missing from it. Therefore  $w$  and the edges on  $w$  can be restored, with each edge  $wv$  ( $v \in V(G \setminus w)$ ,  $d_{G \setminus w}(v) = \Delta(G) - 1$ ) having the colour previously missing at  $v$ .

Let  $P^*$  be the graph obtained from the Petersen graph by deleting one vertex.

Lemma 9.2 With the exception of  $P^*$ , all critical graphs  $G$  of order  $\leq 10$  satisfy the equation.

$$|E(G)| = \left\lfloor \frac{1}{2} |V(G)| \right\rfloor \cdot \Delta(G) + 1.$$

Proof. The result follows by an examination of the list of all critical graphs of order  $\leq 10$  in the papers by Beineke and Fiorini [B1], Jakobsen [J3] and in Chapter 3.

Lemma 9.3 Conjecture 9.1 is true for regular graphs of order  $2n$  and degree  $2n-5$  if  $2n \leq 10$ .

Proof. Let  $w \in V(G)$ . The graph  $G \setminus w$  has four vertices of degree  $\Delta(G)$ , the remainder having degree  $\Delta(G) - 1$ . Therefore

$$\begin{aligned} |E(G \setminus w)| &= \frac{1}{2} (4(2n - 5) + (2n - 5)(2n - 6)) \\ &= 2n^2 - 7n + 5 \\ &< 2n^2 - 7n + 6 \end{aligned}$$

$$= \Delta(G \setminus w) \cdot \left\lfloor \frac{|V(G \setminus w)|}{2} \right\rfloor + 1.$$

By Lemma 9.2,  $G \setminus w$  is not critical, and does not contain a critical subgraph of maximum degree  $2n - 5$  on  $2n - 1$  vertices.

If  $G \setminus w$  is Class 2, it must contain a critical subgraph  $G^*$  of maximum degree  $2n - 5$ . The graph  $G^*$  must satisfy

$$2n - 5 < |V(G^*)| < 2n - 1,$$

and the values  $2n - 4$  and  $2n - 2$  are precluded by Lemma 9.2. Therefore  $G^*$  has  $2n - 3$  vertices and, by Lemma 2.5, three or four vertices of

maximum degree.

There is no solution in this case of the equation of Lemma 9.2 when  $r = 3$ , and the only possibility when  $r = 4$  is that  $G^*$  has four vertices of degree  $2n - 5$ , and the remaining vertices have degree  $2n - 6$ . Since  $G \setminus w$  has only four vertices of maximum degree,  $G \setminus w$  consists of  $G^* \cup K_2$ . Therefore  $d(G) = 2$ , so  $2n - 5 = 2$ , so  $2n = 7$ , which is impossible. Therefore  $G \setminus w$  is Class 1, and so, by Lemma 9.1,  $G$  is Class 1.

### 9.3 Proof of Theorem 9.1

First we prove the special case of Theorem 9.1 when  $d(G) = 2n - 3$ .

#### Case 1. $d(G) = 2n - 3$ .

Let  $w \in V(G)$  and consider the graph  $G \setminus w$ . This has 2 vertices of maximum degree  $2n - 3$  and so, by Lemma 2.5, is Class 1. Therefore, by Lemma 9.1,  $G$  is Class 1.

#### Case 2. $d(G) = 2n - 4$ .

Let  $w \in V(G)$  and consider the graph  $G \setminus w$ . Then  $|V(G \setminus w)| = 2n - 1$ ,  $G \setminus w$  is connected since the conditions imply that  $n \geq 4$ , and  $G \setminus w$  has three vertices of maximum degree. Therefore, by the proposition of Chapter 7,  $G \setminus w$  is Class 1. Therefore, by Lemma 9.1,  $G$  is Class 1.

#### Case 3. $d(G) = 2n - 5$ .

By Lemma 9.3, the theorem is true in this case for  $n \leq 5$ . From now on we shall assume that  $n \geq 6$ .

Let  $w \in V(G)$ . Then  $|V(G \setminus w)| = 2n - 1$ ,  $G \setminus w$  has four vertices, say  $a, b, c, d$ , of maximum degree  $2n - 5$ , and the remaining vertices have degree  $\Delta(G \setminus w) - 1 = 2n - 6$ . By Lemma 9.1 we need only show that  $G \setminus w$  is Class 1.

Suppose  $G \setminus w$  is Class 2. By the proposition of Chapter 7,  $G \setminus w$  could only contain a critical subgraph with the same maximum degree with  $\leq 3$  vertices of maximum degree, if the edge-connectivity of  $G \setminus w$  were  $\leq 1$ . However the minimum degree is too high for this to be possible. Therefore, in any critical subgraph of  $G \setminus w$  with same maximum degree,  $a, b, c$  and  $d$  have degree  $\Delta(G)$ . From Theorems 7.1 and 7.2, we know that the only critical graphs  $G^*$  with four vertices of maximum degree have  $|E(G^*)| = \frac{(|V(G^*)| - 1)}{2} \Delta(G^*) + 1$ . But  $|E(G \setminus w)| = \frac{(|V(G)| - 1)}{2} \Delta(G)$  and hence  $G \setminus w$  is Class 1.

This completes the proof of Theorem 9.1.

#### 9.4 Proof of Theorem 9.2

Proof. Let  $G$  be a regular graph of order  $2n+2$  and degree  $d(G)$  satisfying

$$d(G) \geq \frac{6}{7} |V(G)|.$$

Let  $d(G) = 2n - r + 1$ . By Theorem 9.1, we may assume that  $r \geq 5$ .

Let  $w \in V(G)$  and consider the graph  $G \setminus w$ .  $G \setminus w$  has  $2n + 1$  vertices,  $r$  of degree  $2n - r + 1$  and  $2n - r + 1$  of degree  $2n - r$ . We are going to apply Theorem 8.5 to show that  $G \setminus w$  is Class 1.

First notice that  $\Delta(G \setminus w) = 2n - r + 1 < 2n + 2 - r$ .

Then notice that

$$\delta(G \setminus w) = 2n - r \geq n + \frac{5}{2} r - 1.$$

Finally notice that

$$\begin{aligned} |E(G \setminus w)| &= \frac{1}{2}\{r(2n - r + 1) + (2n - r + 1)(2n - r)\} \\ &= n(2n - r + 1) \\ &= n\Delta \end{aligned}$$

Therefore, by Theorem 8.5,  $G \setminus w$  is Class 1, and so, by Lemma 9.1,  $G$  is Class 1. This proves Theorem 9.2.

We should point out that the following, slightly strengthened form of Theorem 9.2 can be obtained.

Theorem 9.2'. Let  $G$  be a regular graph of order  $2n$  and degree  $2n - k$ , where

$$2n \geq \frac{5}{2} \left\lfloor \frac{(k-1)(2n-k)}{2n-1} \right\rfloor + \frac{9k}{2} - 1.$$

Then  $G$  is Class 1.

This is really a rather insignificant improvement on Theorem 9.2, since the inequality in Theorem 9.2 is approximately  $d(G) \geq .857|V(G)|$ ,

whereas the inequality of Theorem 9.2' is approximately

$d(G) \geq .849|V(G)|$ . Similarly, slight improvements could be made to the inequalities in the theorems of Chapter 8.

We now indicate very briefly how this improvement can be brought about.

In the graph  $G \setminus w$  of the proof of Theorem 9.1, the average value of  $d^*(u)$  ( $u \in V(G \setminus w)$ ) is, by counting edges, easily seen to be

$$\frac{(k-1)(2n-k)}{(2n-1)}$$

Therefore, we can choose the vertex  $v$  (of the proof of Theorem 8.5) so that

$$d^*(v) \leq \left\lfloor \frac{(k-1)(2n-k)}{(2n-1)} \right\rfloor.$$

Now working through the argument of Theorem 8.5 with this bound on  $d^*(v)$  yields a slight improvement to the inequality of that theorem which, in turn, yields the improved bound of Theorem 9.2'.



## 10. Supersnarks

### 10.1 Introduction

In [G1] Martin Gardner gave the name snarks to 3-regular, Class 2 graphs. This name was chosen because of the difficulty of finding such creatures, after Lewis Carroll's "The Hunting of the Snark". At that time it was not known if there were any planar, 3-regular, Class 2 graphs and such an object, he said, would be the mythical Boojum.

A snark is usually defined to be a 3-regular Class 2 graph which is cyclically 4-edge-connected and of girth at least 5. A general discussion and review of known snarks can be found in [C8]. The requirements that snarks should be cyclically 4-edge-connected and of girth at least 5 are made to avoid trivial cases. It now appears [C1] that snarks which are not cyclically 5-edge-connected are 'trivial', in the standard sense that they can be constructed from 3-regular Class 2 graphs of lower order by a standard process. It seems to be premature to try to "define out" such trivial cases.

It is natural to ask whether the idea of a snark can be extended to regular graphs with degree greater than three. One possible generalisation is:

Definition. For  $k \geq 3$ , a k-snark is a regular multi-graph of degree  $k$  and Class 2. If the value of  $k$  is unimportant we use the general term supersnark.

Strictly speaking, a 3-snark is not a snark, as the standard 'trivial cases' have not been excluded. Three trivial cases for  $k$ -snarks are:

- (1) A  $k$ -snark which contains a multiple edge consisting of  $k - 1$  parallel edges. In this case the same colour is forced at either end in any hypothetical  $k$ -edge colouring, so the multiple edge could be contracted out. The converse process also works.

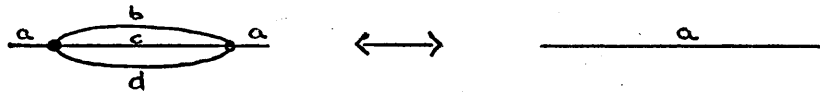


Figure 10.1

- (2) A  $k$ -snark which contains a  $k$ -clique (a complete subgraph on  $k$  vertices) when  $k$  is odd. In this case the outgoing edges must all have different colours in any  $k$ -edge colouring, so the clique could be contracted to a point. The converse process also works.

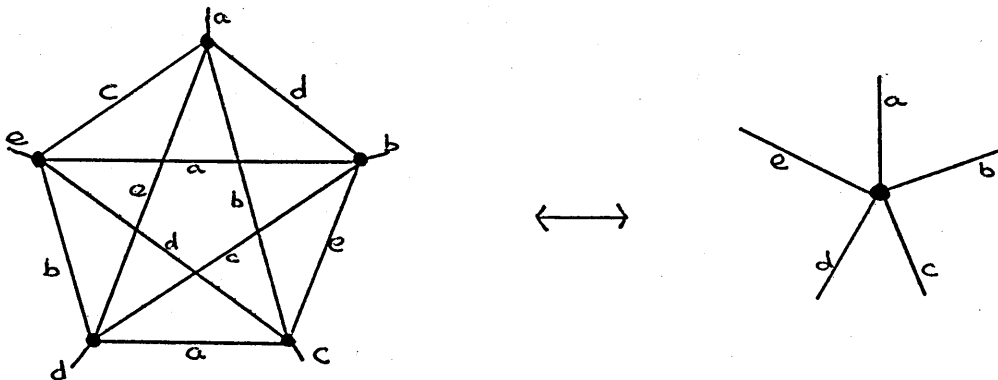


Figure 10.2

- (3) A  $k$ -regular graph of odd order. This must be Class 2 by Lemma 2.7.

Some cases which are certainly trivial when  $k=3$ , and probably are in general (but this still awaits a proof) are given next; the points of difficulty in cases (5) and (6) were overlooked in [C8].

- (4) A  $k$ -snark which contains a  $k$ -clique when  $k$  is even,  $k \geq 4$ .

To discuss this, let  $k$  be even, let  $G$  denote a graph containing a  $k$ -clique, and let  $\mathfrak{D}(G)$  be the set of all derived graphs, where a derived graph is a  $k$ -regular graph obtained from  $G$  by removing the  $k$ -clique leaving  $k$  pendent edges, and then joining these pendent edges together in pairs.

It is clear that if  $G$  is  $k$ -edge-colourable then  $\exists D(G) \in \mathfrak{D}(G)$  such that  $D(G)$  is  $k$ -edge-colourable also (see Figure 10.3).

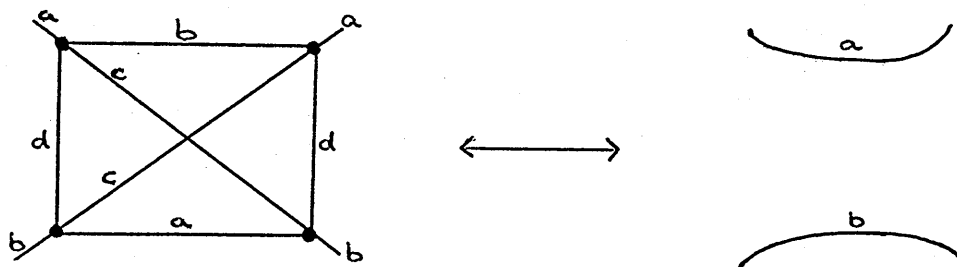


Figure 10.3

On the other hand, except in small cases, it is not clear that if  $G$  is not  $k$ -edge colourable, then  $\exists D(G) \in \mathfrak{D}(G)$  such that  $D(G)$  is not  $k$ -edge-colourable either. This is what needs to be proved if  $k$ -snarks containing  $k$  cliques when  $k$  is even are to be deemed trivial. Putting it another way, we need to know that if  $H \in \mathfrak{D}(G) \Rightarrow H$  is  $k$ -edge-colourable then  $G$  is also  $k$ -edge-colourable.

However, we suspect that much more is true than what we need to prove. We suggest that if  $\exists H \in \mathfrak{D}(G)$  such that  $H$  is  $k$ -edge-colourable, then it follows that  $G$  is  $k$ -edge-colourable. Putting this another way, we suspect that if  $G$  is not  $k$ -edge-colourable, then  $H \in \mathfrak{D}(G) \Rightarrow H$  is not  $k$ -edge-colourable. Rephrasing this one again, we make the following conjecture.

Conjecture 10.1. Let  $k$  be even. Let  $v_1, \dots, v_k$  be the vertices of a  $K_k$  and let  $c_1, \dots, c_k$  be  $k$  colours. Let  $f_1, \dots, f_k \in \{c_1, \dots, c_k\}$  and let  $|\{i \mid f_i = f_j \text{ and } 1 \leq i \leq k\}|$  be even for each  $j$ ,  $1 \leq j \leq k$ . Then  $K_k$  can be properly edge-coloured with  $c_1, \dots, c_k$  in such a way that, for  $1 \leq i \leq k$ , the colour  $f_i$  does not occur on any edge which is incident with  $v_i$ .

There is some possibility that we could prove what we need to deem these  $k$ -snarks trivial without having to prove Conjecture 10.1, but Conjecture 10.1 is probably a tractible problem so tackling it is probably the most sensible approach.

- (5) A  $k$ -snark which contains a  $(k+1)$ -clique with a 1-factor removed, when  $k$  is odd.

Again to discuss this, let  $k$  be odd and let  $G_1$  denote a graph containing a  $(k+1)$ -clique from which a 1-factor has been removed. Let  $\mathfrak{D}(G_1)$  be the set of all graphs derived from  $G_1$ ; here a derived graph is a  $k$ -regular graph obtained from  $G_1$  by removing the  $(k+1)$ -clique from which a 1-factor has been removed, leaving  $k+1$  pendent edges, and then joining these pendent edges together in pairs.

It is clear that if  $G_1$  is  $k$ -edge-colourable, then  $\exists D(G_1) \in \mathfrak{D}(G_1)$  such that  $D(G_1)$  is  $k$ -edge-colourable also (see Figure 10.4).

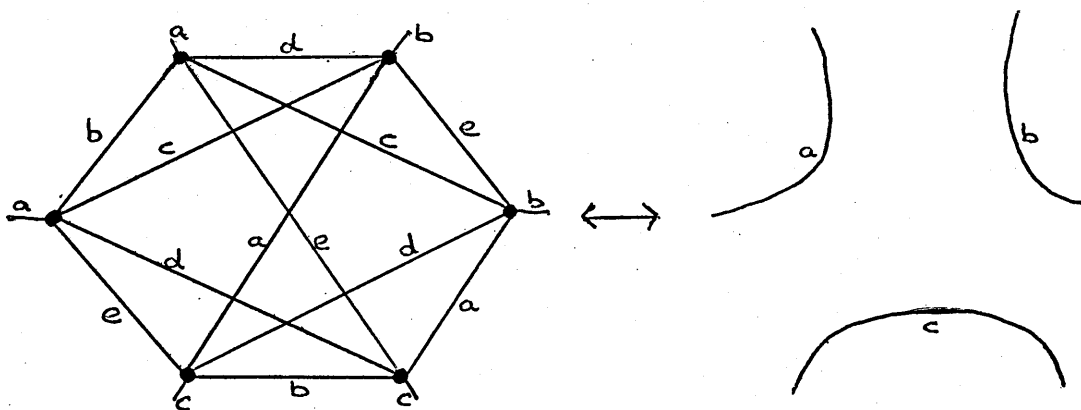


Figure 10.4

On the other hand, except in small cases, it is not clear that if  $G_1$  is not  $k$ -edge-colourable, then  $\exists D(G_1) \in \mathfrak{D}(G_1)$  such that  $D(G_1)$  is not  $k$ -edge-colourable either. Again, this is what needs to be proved for odd  $k$  if  $k$ -snarks containing  $(k+1)$ -cliques from which a 1-factor has been removed are to be deemed trivial. Putting it another way, we need to know that if  $H \in \mathfrak{D}(G) \Rightarrow H$  is  $k$ -edge-colourable, then  $G$  is also  $k$ -edge-colourable.

However, again we suspect that much more is true than what we need to prove. For  $k \geq 5$ , we suspect that if  $\exists H \in \mathfrak{D}(G)$ , such that  $H$  is  $k$ -edge-colourable, then it follows that  $G$  is  $k$ -edge-colourable. [For  $k = 3$ , it seems to be possible that this need not be the case: consider the example of Figure 10.5.

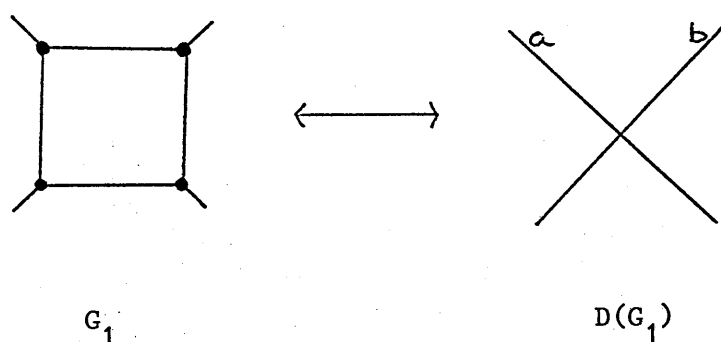


Figure 10.5

If  $D(G_1)$  can only be 3-edge-coloured with the two edges shown receiving different colours, and no other derived graph can be 3-edge-coloured, then  $G_1$  cannot be 3-edge-coloured. However, it is not clear whether this possibility can actually arise. So maybe our suspicion concerning the situation when  $k \geq 5$  should be

extended to the case when  $k = 3$  also]. Putting our suspicion another way, we suspect that, for  $k \geq 5$ , if  $G_1$  is not  $k$ -edge-colourable, then  $H \in \mathfrak{D}(G_1) \Rightarrow H$  is not  $k$ -edge-colourable.

The next conjecture is slightly stronger still.

Conjecture 10.2. Let  $k$  be odd. Let  $F$  be a 1-factor of a  $K_{k+1}$ , let  $v_1, \dots, v_{k+1}$  be the vertices and let  $c_1, \dots, c_k$  be  $k$  colours. Let  $f_1, \dots, f_{k+1} \in \{c_1, \dots, c_k\}$  and let  $|\{i: f_i = f_j \text{ and } 1 \leq i \leq k\}|$  be even for each  $j$ ,  $1 \leq j \leq k$ . Then, apart from one exceptional case,  $K_{k+1} \setminus F$  can be properly edge-coloured with  $c_1, \dots, c_k$  in such a way that, for  $1 \leq i \leq k$ , the colour  $f_i$  does not occur on any edge which is incident with  $v_i$ . The exceptional case is when  $\exists j_1, j_2$  such that, if  $i_1, i_2 \in \{1, \dots, k\} \setminus \{j_1, j_2\}$ , then  $f_{i_1} = f_{i_2}$ , and  $v_{j_1} v_{j_2}$  is an edge of the 1-factor.

If  $D(G_1)$  can be  $k$ -edge-coloured, then  $G_1$  can be given a partial  $k$ -edge-colouring in which all edges are coloured except the edges of the  $K_{k+1}$  less the 1-factor. Let  $f_1, \dots, f_k$  be the colours on the edges connecting  $v_1, \dots, v_k$  to the part of  $G_1$  not in the  $K_{k+1}$  less the 1-factor. Unless we have the exceptional case, then if Conjecture 10.2 is true, this  $k$ -edge-colouring can be extended to all edges of  $G_1$ . In the exceptional case, the partial  $k$ -edge-colouring can be equalized. Then each colour will occur either no times, or two times amongst  $f_1, \dots, f_{k+1}$ . Thus, by recolouring, the exceptional case can be avoided. Note that this argument fails when  $k = 3$ .

Even more than in the last case, in this case it seems that we do not make the problem any simpler by trying to confine our attention just to what we need to prove - the easiest approach seems to be to tackle Conjecture 10.2.

It should be noted that if  $k = 3$ , then the exclusion of (2) and (5) reduces to the requirement that the graph should have girth  $\geq 5$ .

- (6) A  $k$ -snark which contains a  $(k+1)$ -clique with  $\frac{1}{2}(k+1)$  edges removed, when  $k$  is odd.

This is a more general version of (5), and the discussion is very similar.

We can generalise the Parity Lemma [D1] for snarks as follows (this has been used implicitly already in the discussion of (4) and (5) above).

Lemma 10.1 (The generalised Parity Lemma)

Let  $G$  be a  $k$ -regular graph with a  $k$ -edge-colouring. If a cut in  $G$  intersects  $s_i$  edges of colour  $i$ , for  $i = 1, 2, \dots, k$ , and if  $A$  is the set of vertices on one side of the cut, then

$$s_1 \equiv s_2 \equiv \dots \equiv s_k \equiv |V(A)| \pmod{2}.$$



Proof. Consider a graph  $G$  divided by a cut into two sets of vertices  $A$  and  $B$ . Let the cut intersect  $s_i$  edges of colour  $i$ . If  $X$  is the set of edges in the cut, then

$$\sum_{i=1}^k s_i = |X|.$$

If  $A_i$  is the set of edges coloured  $i$  in  $A$  then, since every vertex in  $A$  has an edge coloured  $i$ , where some edges are in  $A_i$  and the rest are in  $X$ ,

$$|V(A)| = 2|A_i| + s_i.$$

Hence  $|V(A)| \equiv s_i \pmod{2}$ , and since this is true for all colours  $i$ ,

$$s_1 \equiv s_2 \equiv \dots \equiv s_k \equiv |V(A)| \pmod{2}.$$

Example. For the following colouring and cut in  $K_5$ , we have

$n_1 = 1, n_2 = 1, n_3 = 1, n_4 = 3, n_5 = 3$  and  $1 \equiv 1 \equiv 1 \equiv 3 \equiv 3 \pmod{2}$ .

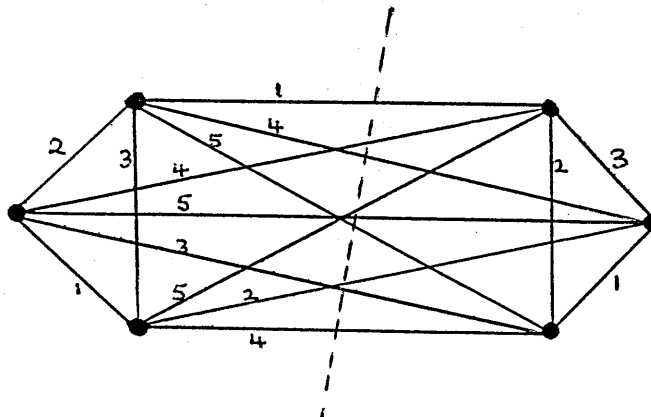


Figure 10.6

Theorem 10.1. Suppose that a  $k$ -snark has a cut set  $C$  of  $r$  edges such that  $G \setminus C$  is  $k$ -edge-colourable. Then either  $r$  is odd and  $r \geq k + 2$ , or  $r$  is even and is at least 4.

Proof. Suppose that  $G$  is an  $r$ -edge-connected  $k$ -snark. Let  $G_1$  and  $G_2$  be two graphs each with  $r$ -semi-edges formed by taking a cut through  $r$  edges of  $G$ .

We assume first that  $r$  is odd and is less than  $k + 2$ . By the Parity Lemma 10.1, if  $G_1$  and  $G_2$  are  $k$ -edge-colourable, then

$$n_{j1} \equiv n_{j2} \equiv \dots \equiv n_{jk} \pmod{2} \text{ and } \sum_{i=1}^k n_{ji} = r,$$

where  $n_{ji}$  is the number of edges of  $G_j$  coloured  $i$ . Since  $r$  is odd,  $\sum_{i=1}^k n_{ji}$  is odd, and hence all the  $n_{ji}$  are odd. But

$$n_{ji} \geq 1 \text{ (}\forall i\text{)} \Rightarrow \sum_{i=1}^k n_{ji} \geq k,$$

so  $r = k$  or  $k + 1$ .

If  $r = k$  then  $n_{ji} = 1$  for all  $i = 1, \dots, k$  and  $j = 1, 2$ , so each graph  $G_j$  has one semi-edge of each colour and, by relabelling the colours of  $G_2$ , these edges will match those of  $G_1$ . Hence  $G$  is Class 1.

This contradiction shows that either  $G_1$  or  $G_2$  is a  $k$ -snark. If  $r = k + 1$  then  $\sum_{i=1}^k n_{ji} = k + 1$ , but this is impossible because  $n_{ji} \geq 1$  and odd.

Now suppose that  $r = 2$ . Then the Parity Lemma cannot be satisfied unless  $k = 2$ . But  $k \geq 4$ , so  $r \geq 4$  also.

## 10.2 Examples of supersnarks

In this section we give a number of examples of supersnarks.

### 10.2.1 Line graphs of 3-snarks

Kotzig [ K2 ] has shown that if  $G$  is a 3-regular graph of order  $n \equiv 0 \pmod{4}$ , then  $G$  is a 3-snark if and only if  $L(G)$  is a 4-snark. Of course, if  $G$  is a 3-regular graph of order  $n \equiv 2 \pmod{4}$  then  $L(G)$  is a 4-snark of a trivial kind, since its order is odd.

If  $G$  is a snark of order  $n \equiv 0 \pmod{4}$ , then its line graph  $L(G)$  is a 4-snark of order  $3n/2$ . The line graphs of Isaacs' flower snarks  $J_3$  and  $J_5$  [I1] are shown in Figures 10.7 and 10.8.

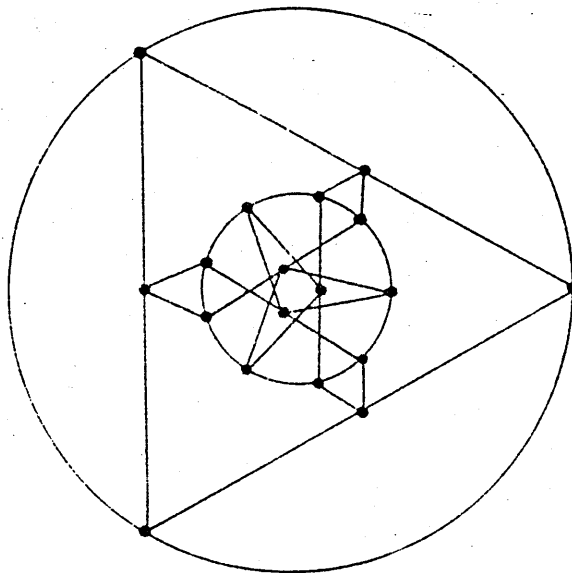


Figure 10.7

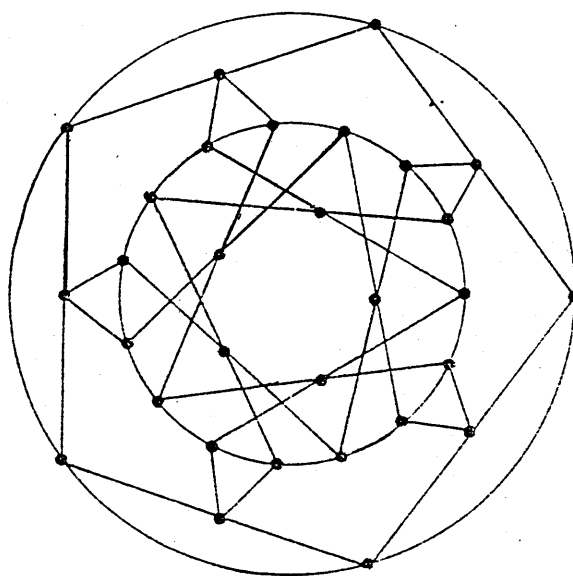


Figure 10.8

### 10.2.2. Generalised line graphs.

In this section we consider a generalisation of Kotzig's result.

Let  $G$  be a 3-regular graph of order  $n$  with vertices  $v_1, \dots, v_n$ . For  $1 \leq i \leq n$ , let  $H_i$  be a graph with three vertices of degree 2 and the remaining vertices of degree 4. Let  $|V(H_i)| = h_i$  ( $1 \leq i \leq n$ ). Form a graph from  $G, H_1, \dots, H_n$  by identifying one vertex of  $H_i$  of degree 2 with one vertex of  $H_j$  of degree 2 whenever the edge  $v_1 v_2$  is an edge of  $G$ , using each vertex of degree 2 exactly once. Let  $G(H_1, \dots, H_n)$  denote a graph formed this way; then  $G(H_1, \dots, H_n)$  is 4-regular. If  $h_i = 3$  ( $1 \leq i \leq n$ ), then  $G(H_1, \dots, H_n)$  is the line graph of  $G$ .

We prove the following two theorems on generalised line graphs.

Theorem 10.2. If  $G$  is Class 2, then  $G(H_1, \dots, H_n)$  is Class 2.

Theorem 10.3. Let  $h_i$  be odd ( $1 \leq i \leq n$ ). Then

$G(H_1, \dots, H_n)$  is Class 2.

if and only if

either  $h_1 + \dots + h_n - \frac{3n}{2}$  is odd,

or one of  $H_1, \dots, H_n$  is Class 2,

or  $G$  is Class 2.

We do not know whether Theorem 10.3 holds when the restriction that  $h_i$  be odd ( $1 \leq i \leq n$ ) is removed.

Proof of Theorem 10.2. Assume that  $G(H_1, \dots, H_n)$  is Class 1. Then we prove the theorem by showing that  $G$  is Class 1.

Let  $G(H_1, \dots, H_n)$  be edge-coloured with four colours  $a, b, c, d$ . From  $G(H_1, \dots, H_n)$ , form a new graph  $G^*$  as follows: replace each subgraph  $H_i$  by a vertex joined by two edges to each of the three vertices of  $H_i$  of degree 2, keeping the colours as shown.

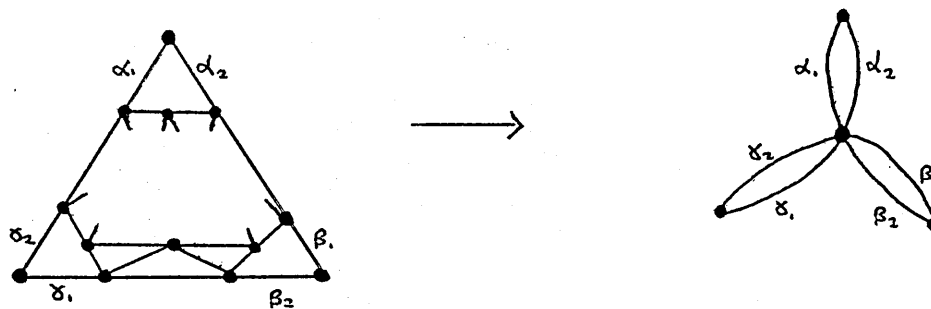


Figure 10.9

From  $G^*$  we obtain  $G$  by removing each vertex of degree 4 and replacing the pair of double edges on the vertex by a single edge:

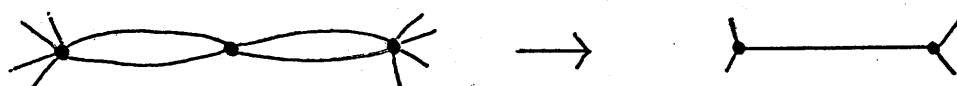


Figure 10.10

When we remove these pairs of double edges, we put a corresponding colour on the replacement edge, as indicated:

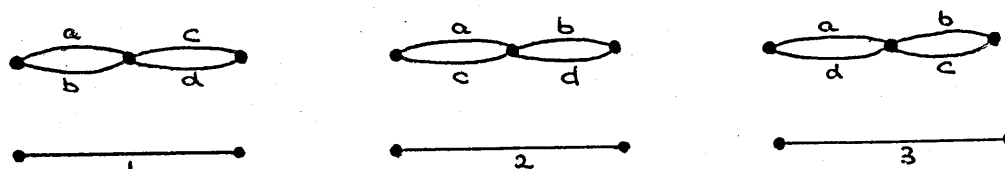


Figure 10.11

Now each edge of  $G$  is coloured with one of the colours 1, 2, 3. We can see that this is a proper colouring of  $E(G)$  since, if a vertex  $v_i$  in  $G$  has more than one edge of any colour, then the corresponding subgraph  $H_i$  of  $G(H_1, \dots, H_n)$  must have had two of its vertices of degree 2 coloured with exactly the same colours, or no colours the same.

But by the Parity Lemma applied to the 6 (or fewer) edges joining the three vertices of degree 2 in  $H_i$  to the rest of  $H_i$ , we see that this is not possible. Hence  $G$  is Class 1.

### Proof of Theorem 10.3

Sufficiency. If  $G$  is Class 2, then, by Theorem 10.2,  $G(H_1, \dots, H_n)$  is Class 2. If  $h_1 + \dots + h_n - \frac{3n}{2}$  is odd, then  $|V(G(H_1, \dots, H_n))|$  is odd, so, by Lemma 2.6,  $G(H_1, \dots, H_n)$  is Class 2. Finally, if one of  $H_1, \dots, H_n$  is Class 2, then, since  $H_1, \dots, H_n$  are subgraphs of  $G(H_1, \dots, H_n)$  of degree 4, it follows that  $G(H_1, \dots, H_n)$  is Class 2.

Necessity. Suppose that  $h_1 + \dots + h_n - \frac{3n}{2}$  is even, that  $H_1, \dots, H_n$  are all Class 1, and that  $G$  is Class 1. We prove the necessity by showing that  $G(H_1, \dots, H_n)$  is Class 1.

Since  $n$  is even and  $h_1, \dots, h_n$  are all odd, it follows that  $h_1 + \dots + h_n$  is even. Consequently, if it were true that  $n \equiv 2 \pmod{4}$ , then  $h_1 + \dots + h_n - \frac{3n}{2}$  would be odd, a contradiction. Therefore,  $n \equiv 0 \pmod{4}$ . Consequently, by Kotzig's result,  $L(G)$  is Class 1.

Since  $h_i$  is odd and  $H_i$  is Class 1 ( $1 \leq i \leq n$ ), it follows from the Parity Lemma that  $H_i$  can be 4-edge-coloured with the vertices of degree two having their edges coloured with the pairs of colours  $\{1,2\}, \{2,3\}, \{3,1\}$ . Consequently, in any 4-colouring of  $L(G)$ , a triangle of  $L(G)$  corresponding to a vertex  $v_i$  of  $G$  can be substituted for by  $H_i$ , the three colours on the edges of the triangle being compatible with the edge-colouring of  $H_i$ . Thus  $G(H_1, \dots, H_n)$  is Class 1.

### 10.2.3 Petersen multigraph $k$ -snarks

In this section and the next we give two ways of forming  $k$ -snarks, for  $k > 3$ .

In this section we consider a set of  $k$ -regular multigraphs based on the Petersen graph and show which of these are Class 2. Then we show that from these multigraphs one may construct  $k$ -regular graphs of the same Class.

We consider the Petersen multigraph  $M_{r,s}$  shown below,

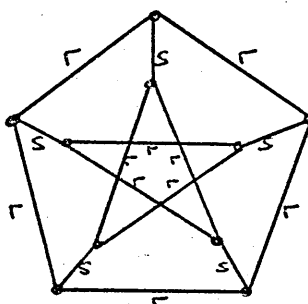


Figure 10.12

where the edges labelled  $r, s$  have  $r$  and  $s$  multiple edges respectively. Meredith [M2] has shown that for  $r = 2k + 1$ ;  $s = 2k - 3, 2k - 2, 2k - 1, 2k, 2k + 1, 2k + 2$ , the graph  $M_{r,s}$  is Class 2 and for  $r = 2k$ ;  $s = 2k, 2k + 1, 2k + 2$  the graph  $M_{r,s}$  is Class 1.

Theorem 10.4. If  $r$  is odd or if  $r$  is even and greater than  $2s$ , then the graph  $M_{r,s}$  is Class 2; otherwise the graph is Class 1.

Proof. We first consider the 1-factors of the Petersen graph. Let the inside edges, outside edges and spoke edges have the obvious meanings, as indicated in the diagram.



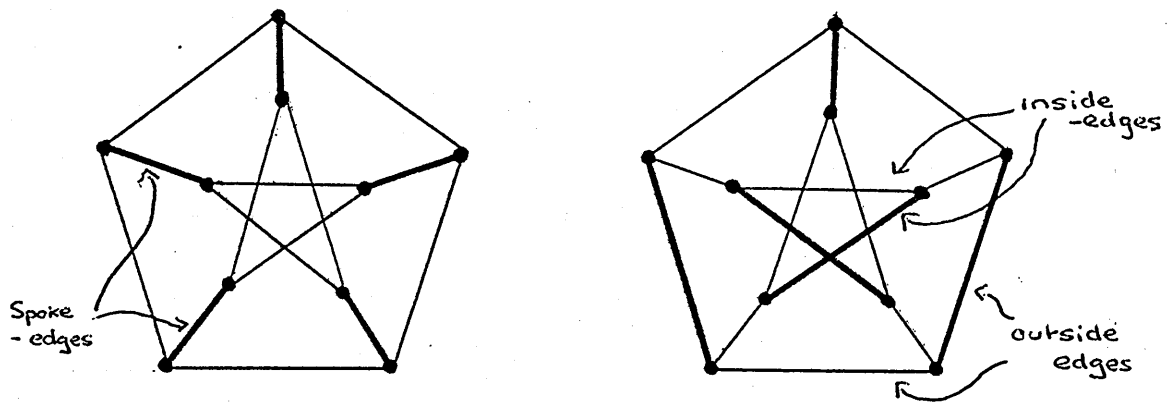


Figure 10.13

Either all 5-spokes are used, or only one is used, in a 1-factor. Hence the only possible types of 1-factor are as shown in Figure 10.13.

We consider the following three cases:

$r$  odd. If  $r = 2k + 1$ , then the number of outside edges is

$$5r = 5(2k + 1) = 10k + 5.$$

Now any 1-factor has an even number of outside edges. Hence, after taking out  $r$  1-factors, we have used up an even number of outside edges. Since the number of outside edges is odd,  $M_{r,s}$  cannot have a 1-factorisation. Hence if  $r$  is odd, then  $M_{r,s}$  is Class 2.

$r$  even and  $r > 2s$ . If  $M_{r,s}$  has a 1-factorisation then, whenever two outside edges are in a 1-factor, one spoke is in the same 1-factor. Since there are  $5r$  outside edges, the number of spokes is  $\geq \frac{5}{2}r$ , so  $5s \geq \frac{5}{2}r$ ; that is  $2s \geq r$ . Since  $r > 2s$  we have a contradiction. Hence  $M_{r,s}$  does not have a 1-factorisation and is therefore of Class 2.

$r$  even and  $r \leq 2s$ . In this final case we show that there is a 1-factorisation. Consider the 1-factors which use just one spoke, and take five of these, each containing a different spoke. These five 1-factors cover each spoke once and every inner and outer edge twice.

If we take  $\frac{1}{2}r$  sets of these five 1-factors, we will have included each inner and each outer edge in a 1-factor. The remaining edges will be spokes. Each spoke is in  $\frac{1}{2}r$  1-factors. We put the remaining spokes into  $s - \frac{1}{2}r$  1-factors, where these 1-factors are the ones which use all spokes.

We have now put  $M_{r,s}$  into  $\frac{5}{2}r + s - \frac{1}{2}r = s + 2r$  1-factors.

If we give the edges of each 1-factor a different colour, then we have coloured  $M_{r,s}$  with  $s + 2r$  colours, and hence  $M_{r,s}$  is Class 1.

This proves Theorem 10.4.

We now show how to obtain from a multigraph  $M_{r,s}$  a simple regular graph with degree  $2r + s$  of the same Class. This is easily done. We replace each multiple edge  $uv$  consisting of  $r$  parallel edges by a Class 1 graph with  $2r$  vertices of degree  $2r + s - 1$ , the remaining vertices being of degree  $2r + s$ . When such a graph is properly edge-coloured with  $2r + s$  colours, each colour is missing from exactly two vertices. For each such colour, join one of these vertices to  $u$  and one to  $v$ . The process is indicated in Figure 10.14:

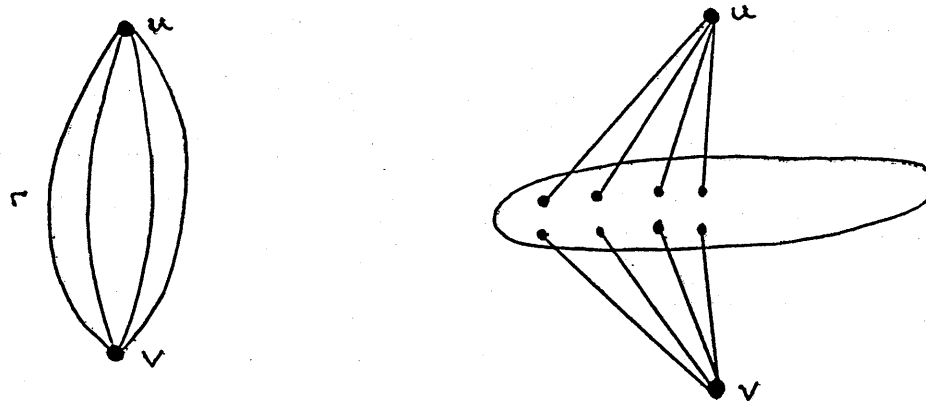


Figure 10.14

A similar process works for the other type of multiple edge. It is easy to see that  $M_{r,s}$  is Class 1 if and only if any corresponding simple graph obtained by this process is Class 1.

An example of the process described above when  $r = 2$ ,  $2r + s = 5$ .

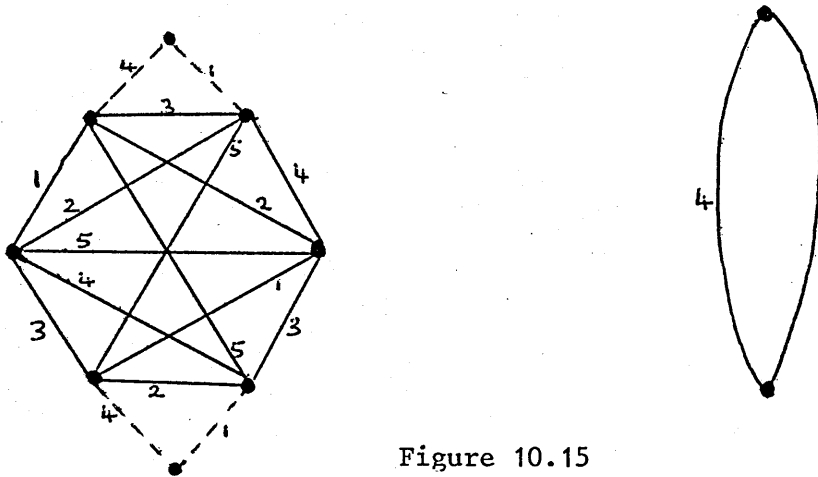


Figure 10.15

An example of a 4-snark constructed by this process (here  $r = 1$ ,  $s = 2$ ).

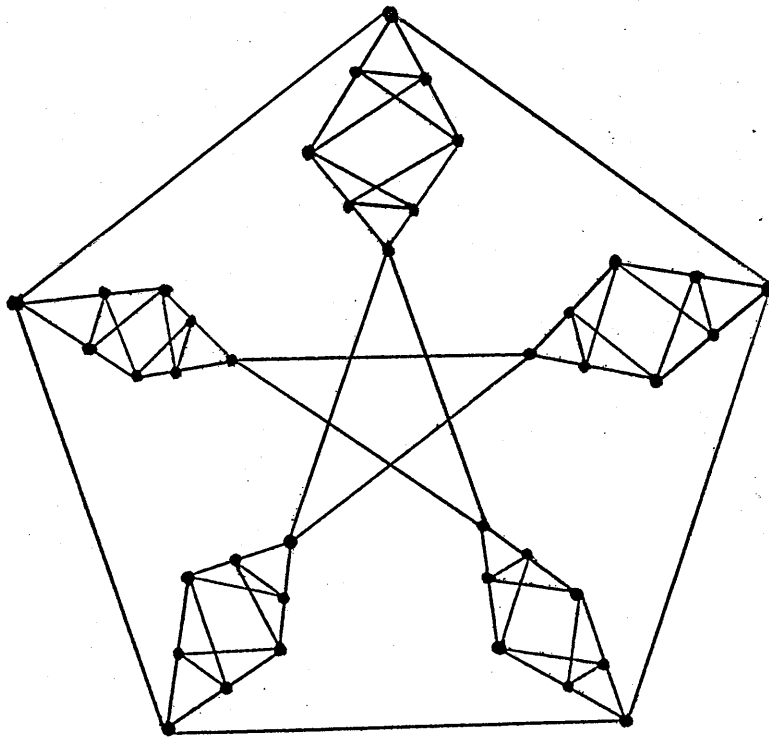


Figure 10.16

Another construction, used by Meredith [ M2 ], will suffice to obtain from  $M_{r,s}$  a simple regular graph of degree  $2r + s$  of the same Class. It can be used instead of, or together with, the previous construction. In this construction we replace each vertex of  $M_{r,s}$ , or any of the corresponding vertices of a graph obtained from  $M_{r,s}$  by the previous construction, by the complete bipartite graph  $K_{2r+s, 2r+s-1}$ , as shown in Figure 10.17.

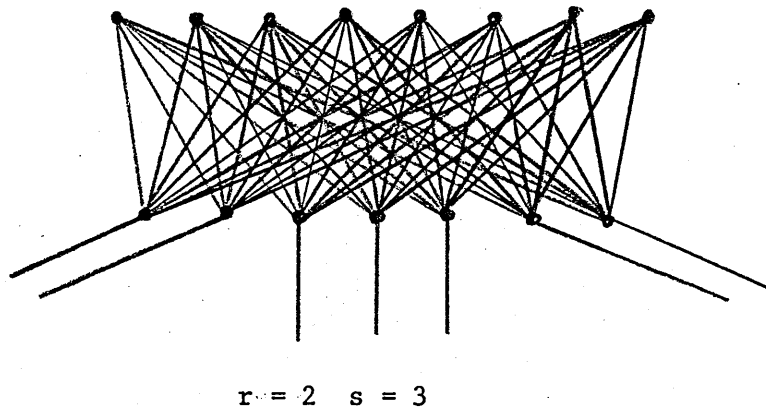


Figure 10.17

Any colouring of  $K_{2r+s, 2r+s-1}$  with  $2r+s$  colours will mean that each of the  $2r+s$  vertices of degree  $2r+s-1$  will have exactly one colour missing, and these missing colours will all be different. Hence colouring  $M_{r,s}$  is equivalent to colouring the graph obtained from  $M_{r,s}$  by these replacements.

An example (with  $r = 1$ ,  $s = 2$ ) of a 4-snark obtained by Meredith [M2] by this second process.

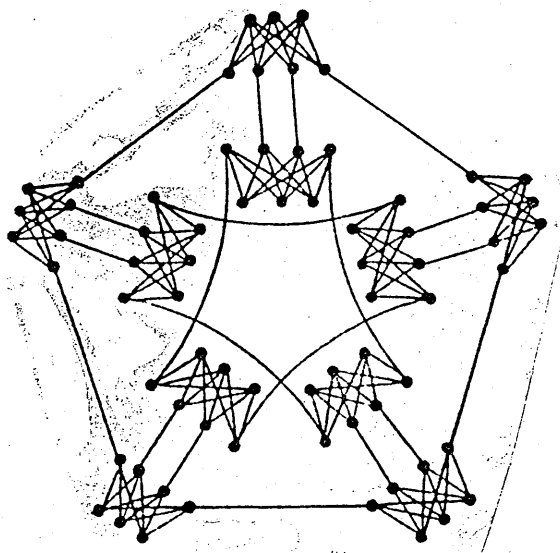


Figure 10.18

#### 10.2.4 Another family of $k$ -snarks

Here we define a family  $M(J)$  of  $k$ -snarks based on the first flower snark  $J_3$  and a particular one of its 1-factors  $F$ . The first flower snark  $J_3$ , and the particular 1-factor we require, are illustrated in Figure 10.19.

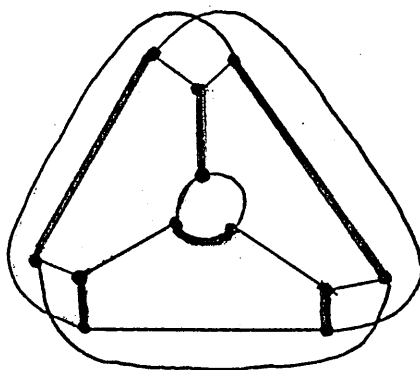


Figure 10.19

$M(J)$  is formed by replacing each edge of  $F$  by  $k - 2$  parallel edges.

Clearly  $M(J)$  is a regular multigraph of degree  $k$ .

Theorem 10.5.  $M(J)$  is Class 2.

Proof. Assume  $M(J)$  is Class 1; then  $M(J)$  has  $k$  1-factors. There are two kinds of 1-factor in  $M(J)$ , as illustrated:

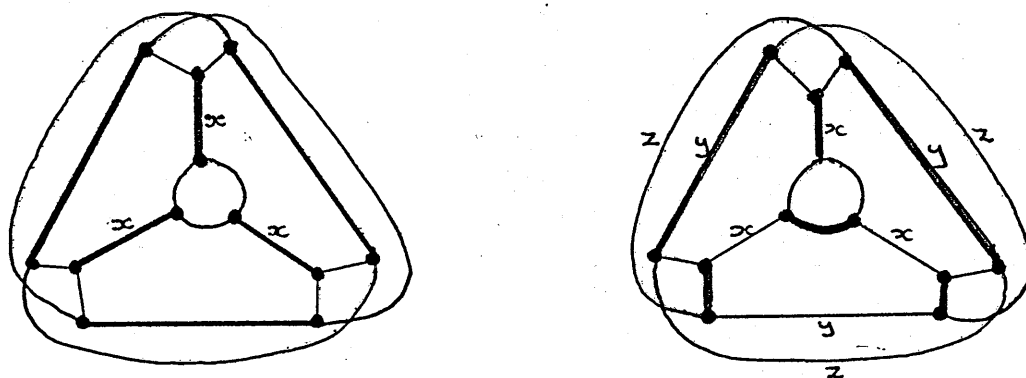


Figure 10.20

Every 1-factor uses either 3 edges or 1 edge of type  $x$ . There are exactly  $k - 2 + 2 = k$  edges of type  $x$  in  $M(J)$ , so the  $k$  1-factors must each contain exactly one edge of type  $x$ . These 1-factors will contain either 0 or 2 edges of type  $y$ . There are  $2(k-2) + 1 = 2k - 3$  type  $y$  edges. Since we cannot cover an odd number of edges with edge-disjoint 1-factors, we have a contradiction. Therefore  $M(J)$  is Class 2.

Either or both of the constructions described in the previous section can be used to replace the multiple edges in  $M(J)$ , and to yield simple graphs which are  $k$ -snarks.

### 10.3 The classification of all regular graphs of order at most 10.

Here we give a complete list of all regular Class 2 graphs of even order at most 10. These results were obtained using a list of all graphs on up to 10 vertices and running an edge-colouring programme on each graph; we only considered even order graphs, since all regular graphs of odd order are Class 2.

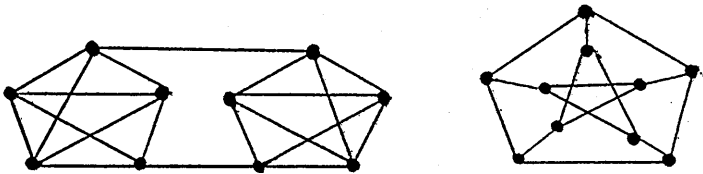
Order	Total number of graphs	Class 2 graphs
4	2	
6	6	$C_3 + C_3$
8	19	$C_5 + C_3$
10	172	$C_7 + C_3, C_5 + C_5, K_5 + K_5,$
		

Figure 10.21

**Definition.** A  $k$ -snark  $G$  is proper if for any Class 2 subgraph  $G'$  with maximum degree  $k$ , there does not exist a  $k$ -snark  $G^*$  containing  $G'$  of smaller order.

From these results we can see that the Petersen graph is the only proper simple  $k$ -snark on at most 10 vertices. We now give another proof of this result using the results of Chapter 3 on critical graphs.

Theorem 10.6. The Petersen graph is the only proper simple  $k$ -snark on at most 10 vertices.

Proof. We have seen from Chapter 3 that, apart from the 3-critical graph  $H$  of the figure below, the only  $k$ -critical graphs on at most 10 vertices have  $2n + 1$  vertices and  $kn + 1$  edges. The only graphs of order at most 10, which have  $2n + 1$  vertices and  $kn + 1$  edges, which are not  $k$ -critical are those formed by the addition of an edge joining two of the vertices of degree 2 of  $H$ .

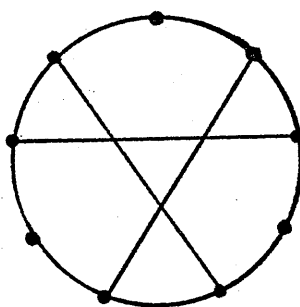


Figure 10.22

By Lemma 2.8, if  $G$  is a proper simple  $k$ -snark on at most 10 vertices, then  $G$  contains a  $k$ -critical graph on at most 10 vertices.

If  $k = 3$  and  $G$  contains the graph  $H$ , then  $G$  must be a 3-snark, and the only way to make  $H$  into a 3-regular simple graph on at most 10 vertices is to form the Petersen graph.

In any other case,  $G$  has a  $k$ -critical simple subgraph  $G^*$  of order  $2n^* + 1$ , whose deficiency is

$$\begin{aligned} &= k(2n^* + 1) - 2(kn^* + 1) \\ &= k - 2. \end{aligned}$$



The number of edges needed to construct a  $k$ -regular simple graph  $G$  from  $G^*$  is

$$\frac{1}{2} \{k - 2 + k(|V(G)| - |V(G^*)|)\}.$$

However, without forming multiple edges, not more than

$$(k - 2) + \frac{(|V(G)| - |V(G^*)|)}{2}$$

edges can be placed on these vertices. So if  $G^*$  can be extended to  $G$ , then

$$\frac{1}{2}(k - 2) + \frac{1}{2}k(|V(G)| - |V(G^*)|) \leq (k - 2) + \frac{(|V(G)| - |V(G^*)|)}{2},$$

so

$$k \leq |V(G)| - |V(G^*)| - 1.$$

Therefore

$$k \leq \frac{1}{2} |V(G)| - (k + 1)$$

from which it follows that

$$2k + 2 \leq |V(G)|.$$

Since  $k \geq 4$  and  $|V(G)| \leq 10$ , the only possibility for  $G^*$  has

$$|V(G^*)| = 5 \text{ and } k = 4$$

but then  $G^*$  is a subgraph of  $K_5$  and  $G$  is not proper.

#### 10.4 The Generalised Double-Star Snark

The Petersen graph may be obtained by taking an outer 5-cycle with a semi-edge on each vertex (called a spoke) and an inner 5-cycle attached by joining its vertices to every second spoke. M. E. Watkins defined the generalised Petersen graph  $P(n, k)$  as follows: The graph  $P(n, k)$  has vertices  $v_0, v_1, \dots, v_{n-1}, v'_0, v'_1, \dots, v'_{n-1}$  and edges  $v_i v_{i+1}$ ,  $v'_i v'_{i+k}$  and  $v_i v'_i$  for all  $i$  with  $0 \leq i \leq n - 1$ , with all subscripts taken

modulo  $n$ . Thus the Petersen graph is the graph  $P(5,2)$ . Watkins [W1] conjectured that with the single exception of  $P(5,2)$ , all of these graphs are Class 1. In 1973 the conjecture was settled in the affirmative by Castagna and Prins [C2].

In a similar way we can define the generalised double-star snark, and prove the following:

Theorem 10.7. The generalised double-star snark  $D(n,k)$ ,  $k \leq \frac{1}{2}n$  is Class 2 if and only if  $D(n,k)$  is one of  $D(1,1)$ ,  $D(3,1)$ ,  $D(5,2)$  or  $D(n,k)$ , where  $\frac{n}{3}$  is an integer and  $\text{g.c.d.}(n,k) = \frac{n}{3}$ .

Before giving the definition of a double star snark, we give an example:

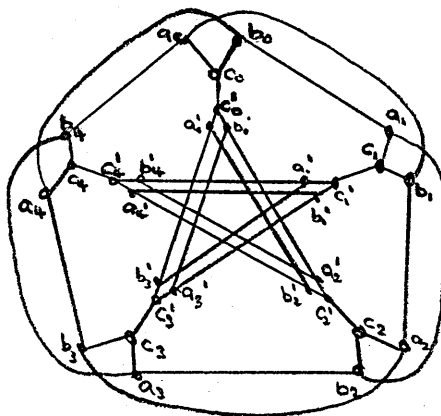


Figure 10.23

$D(5,2)$  is the double-star snark discovered by Isaacs.

Definition. Let  $n$  and  $k$  be positive integers,  $1 \leq k \leq n$ . The generalised double star snark has  $6n$  vertices denoted by

$$a_0, a_1, \dots, a_{n-1}, b_0, b_1, \dots, b_{n-1}, c_0, c_1, \dots, c_{n-1},$$

$$a'_0, a'_1, \dots, a'_{n-1}, b'_0, b'_1, \dots, b'_{n-1}, c'_0, c'_1, \dots, c'_{n-1},$$

and edges

$$a_i c_i, b_i c_i, b_i a_{i+1}, a_i b_{i+1},$$

$$a'_i c'_i, b'_i c'_i, b'_i a'_{i+k}, a'_i b'_{i+k},$$

$$c_i c'_i \text{ for all } i \text{ with } 0 \leq i \leq n-1, \text{ with all subscripts taken modulo } n.$$

The sets of edges  $\{a_i c_i, b_i c_i, b_i a_{i+1}, a_i b_{i+1}\}$ , and  $\{a'_i c'_i, b'_i c'_i, b'_i a'_{i+k}, a'_i b'_{i+k}\}$  make up the inner and outer rims respectively.



Figure 10.24

The edges  $c_i c'_i$  are called spokes.

In order to see that a graph is Class 1, we look for 2-factors where each cycle has even length. The 2-factors can then be 2-coloured and the remaining edges will form a 1-factor which can be coloured with the third colour, and hence the graph will be Class 1.

It is easy to see by symmetry that  $D(n, k) \cong D(n, n-k)$  and hence we need only consider  $k$  such that  $1 \leq k \leq \frac{1}{2}n$ . We now introduce some terminology to describe various paths and cycles that will be used in

the following constructions of 2-factors.

$L_{jk}$  is the path indicated in the outer rim from  $c_j$  to  $c_k$ :

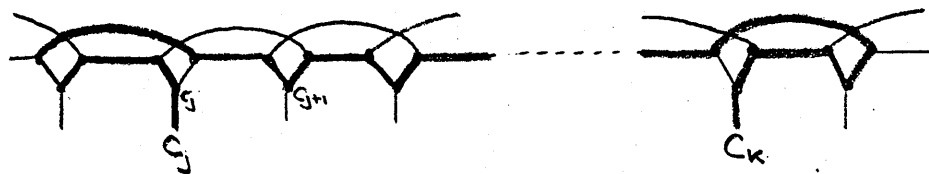


Figure 10.25

$L'_{jk}$  is the similar path in the inner.

$M_{jk}$  is the path indicated in the outer rim from  $c_j$  to  $c_k$ :

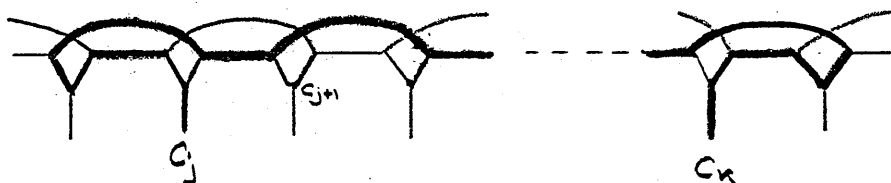


Figure 10.26

$M'_{jk}$  is the similar path in the inner.

$N_{a,b,c,d}$  is the mixture indicated of these two types:

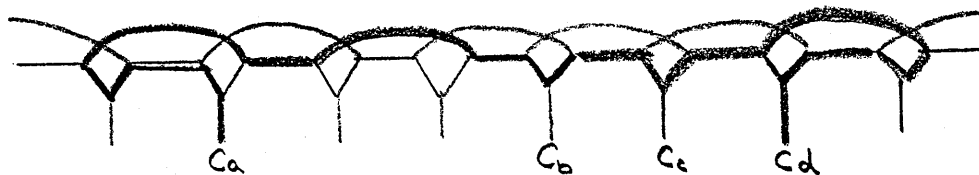


Figure 10.27

$N_{a,b,c,d,e}$  is a similar mixture of the two types:

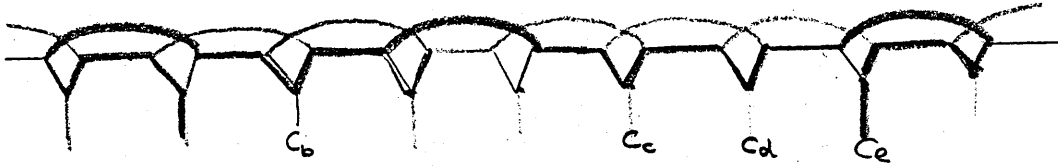


Figure 10.28

$P_j$  is the path indicated of length 3:

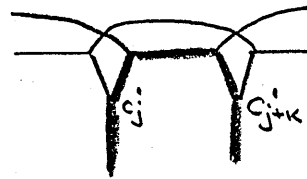
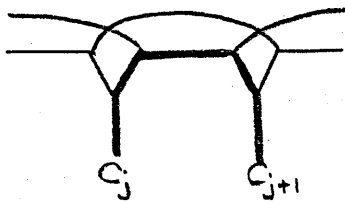


Figure 10.29

$O_j$  is the circuit indicated of length 6:

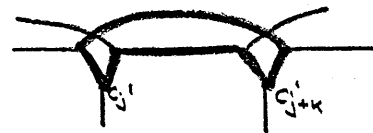
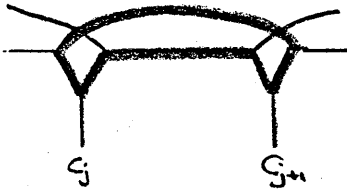


Figure 10.30

$I_i$  is the component in the inner rim containing the vertices  $a_i'$ ,  $b_i'$ ,  $c_i'$ .

Finally we remark that we take the length of a path to be the number of edges it contains.

Proof of Theorem 10.7. We shall use the standard usage that  $(n,k)$  is the greatest common divisor of  $n$  and  $k$ .

Case 1.  $(n,k) = 1$  and  $n$  is even. Then  $D(n,k)$  is Class 1. We form a 2-factor consisting of two even cycles, one around the outer rim of length  $3n$

$$(a_0 c_0 b_0 a_1 c_1 b_1 \dots a_{n-1} c_{n-1} b_{n-1} a_0)$$

and the corresponding cycle around the inner rim. Hence  $D(n,k)$  is Class 1.

Example.  $D(4,1)$ .

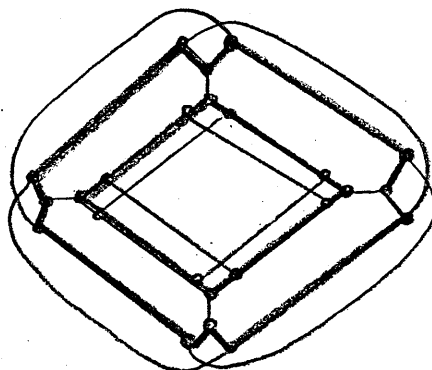


Figure 10.31

Case 2.  $(n,k) = 1$ ,  $n$  is odd and  $n \leq 5$ . There are four graphs to consider,  $D(1,1)$ ,  $D(3,1)$ ,  $D(5,1)$  and  $D(5,2)$ .

$D(1,1)$ :



This graph has multiple edges and is of Class 2.

Figure 10.32

$D(3,1)$ :

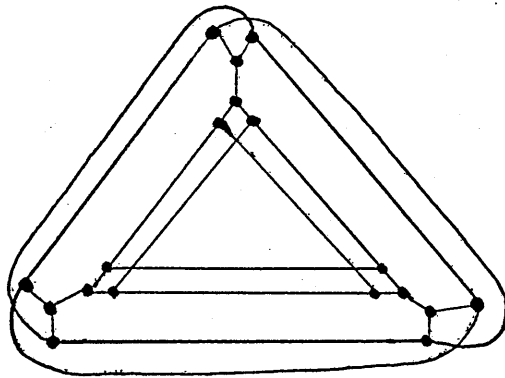


Figure 10.33

This graph is Class 2. The inner rim is isomorphic to the 3-critical graph on 9 vertices contained in the Petersen graph.

$D(5,1)$ :

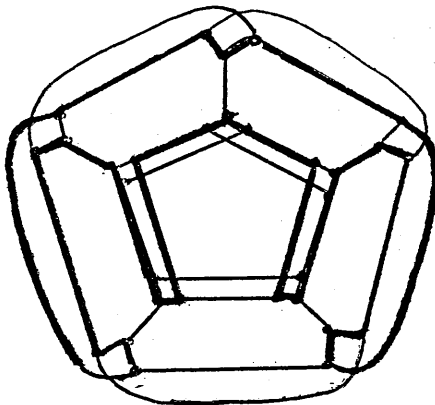


Figure 10.34

This graph has a Hamiltonian cycle as shown, and hence is Class 1.

$D(5,2)$ : This is the double-star snark as drawn in Figure 10.23 and is Class 2.

Case 3.  $(n,k) = 1$ ,  $n$  is odd and at least 7 and  $k = 2$ . We construct a path of odd length  $3n$  in the outer rim and join it up with a path of odd length in the inner rim:

The outer path is from  $c_1$  to  $c_{n-2}$  and is  $L_{1,n-2}$ .

The inner path is  $\begin{cases} L'_{n-2,1} & \text{if } n = 4t + 1, \\ L'_{1,n-2} & \text{if } n = 4t + 3. \end{cases}$

In both cases the two paths join to give a cycle of even length. We now find cycles of length 6 in the inner rim to cover the remaining vertices.

The cycles are

$$\begin{cases} 0'_5, 0'_9, \dots, 0'_{n-8} & \text{if } n = 4t + 1, \\ 0'_2, 0'_6, \dots, 0'_{n-5} & \text{if } n = 4t + 3. \end{cases}$$

Each vertex is in some even cycle and hence  $D(n,k)$  is Class 1.

Example:  $D(9,2)$ .

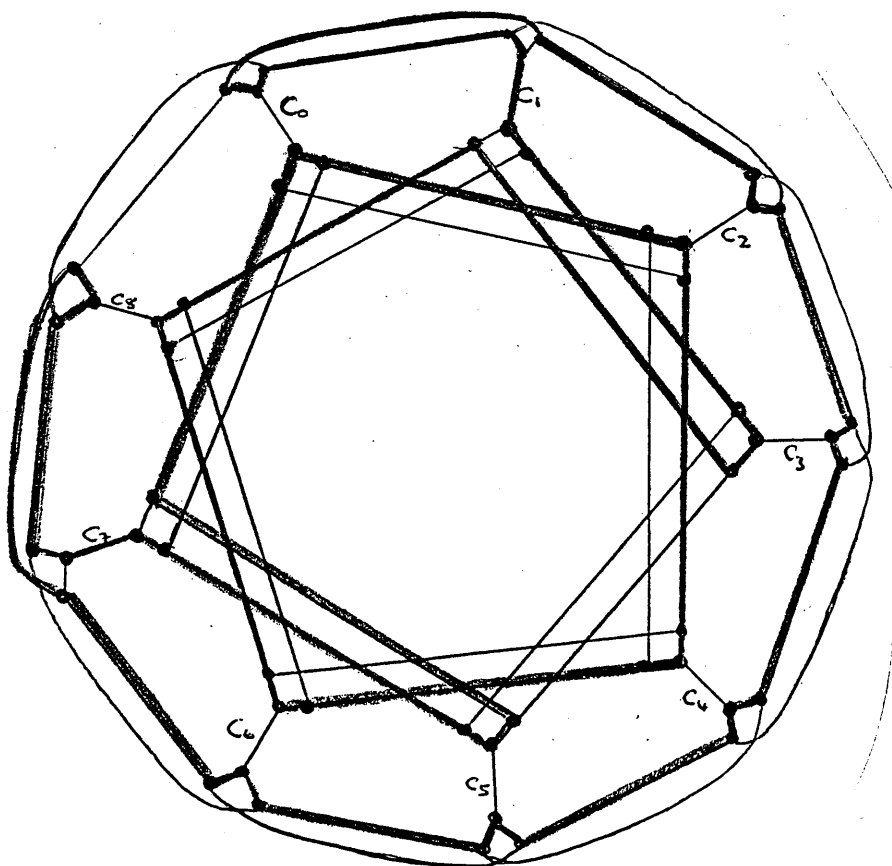


Figure 10.35



Case 4.  $(n,k) = 1$ ,  $n$  is odd and at least 7 and  $k \neq 2$ . We find an odd length path in the outer rim and join it to an odd length path in the inner rim. The remaining vertices on the outer and inner rims respectively are covered by even cycles of length six.

Our odd length path in the outer rim will be  $L_{n-1,1}$ , the vertices not on the path are covered by the cycles  $0_3, 0_5, \dots, 0_{n-4}$ .

Next we find a path of odd length in the inner rim from  $c_1$  to  $c_{n-1}$ . Either  $L'_{n-1,1}$  or  $L'_{1,n-1}$  has odd length. To see this, note that  $L'_{1,n-1}$  covers vertices with subscripts

$$1 - k, 1, k, \dots, 1 + xk = n - 1, n - 1 + k,$$

for some  $x$ , and  $L'_{n-1,1}$  covers vertices with subscripts

$$n-1-k, n-1 = 1+xk, n-1+k, \dots, n-1+yk = 1,$$

where  $n - 1 + yk = 1 + (x + y)k = 1 + nk$ , so that  $x + y = n$ , so that one of  $x$  and  $y$  is odd, the other even. Therefore one of  $L'_{n-1,1}$  and  $L'_{1,n-1}$  has odd length. There will be an even number of vertices on the inner rim not on the path. They can be covered by cycles  $0_j$ .

We now have a 2-factor of  $D(n,k)$  with lots of 6 cycles and a long even length cycle using vertices from both the inner and the outer rims. Hence  $D(n,k)$  is Class 1.

We now give an example to show how the 2-factor is found in the inner rim:

The inner rim of  $D(13,4)$ .

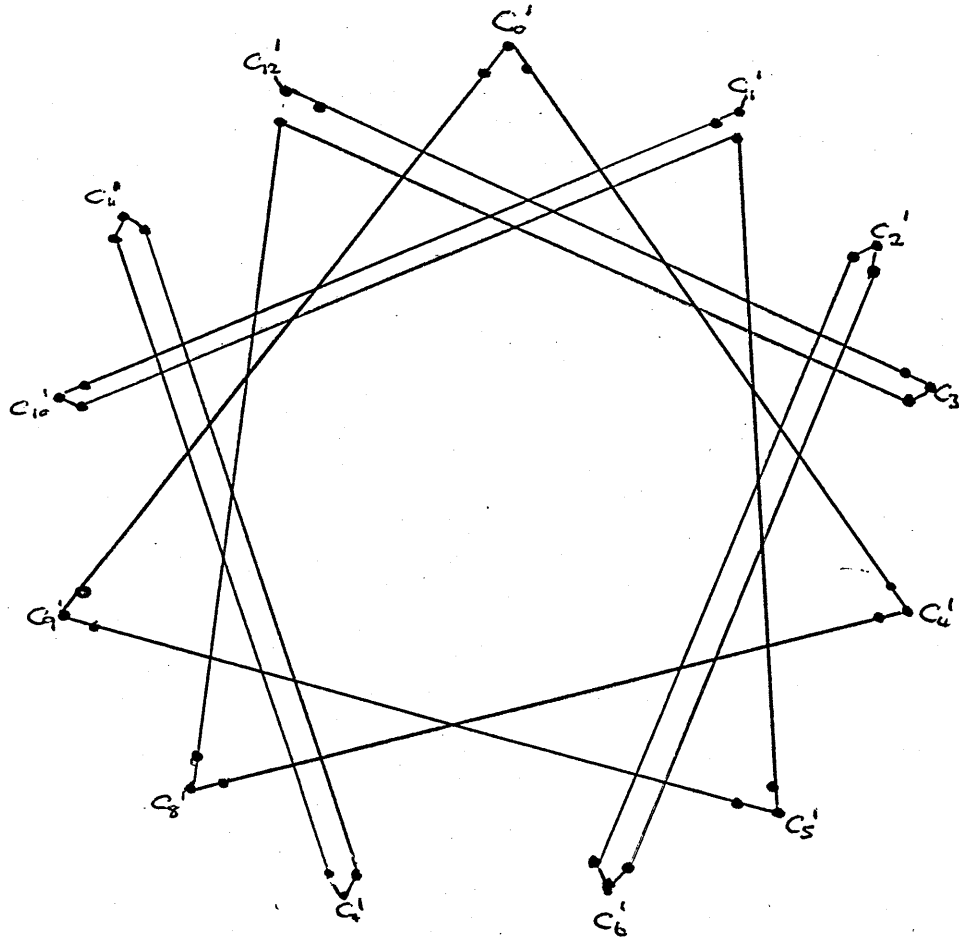


Figure 10.36

The odd path is  $L'_{1,n-1}$  and there are two 6-cycles,  $0'_7$  and  $0'_2$ .

Case 5.  $n/(n,k)$  is even. Then  $n$  must be even. Therefore the inner rim of  $D(n,k)$  will be composed of  $(n,k)$  components each with  $\frac{n}{(n,k)}$  spokes. We can find one even cycle in the outer rim and one even cycle in each of the inner components, and these will cover all vertices of  $D(n,k)$ .

The outer cycle of length  $3n$  is

$$(a_0 c_0 b_0 \dots a_{n-1} c_{n-1} b_{n-1}, a_0).$$

The inner cycles of length  $3n/(n,k)$  for each  $i$ ,  
 $0 \leq i \leq (n,k) - 1$ , are

$$(a_i^! c_i^! b_i^! a_{i+k}^! c_{i+k}^! b_{i+k}^! \dots a_{i-k}^! c_{i-k}^! b_{i-k}^! a_i^!).$$

Hence  $D(n,k)$  is Class 1.

Example:  $D(8,2)$ .

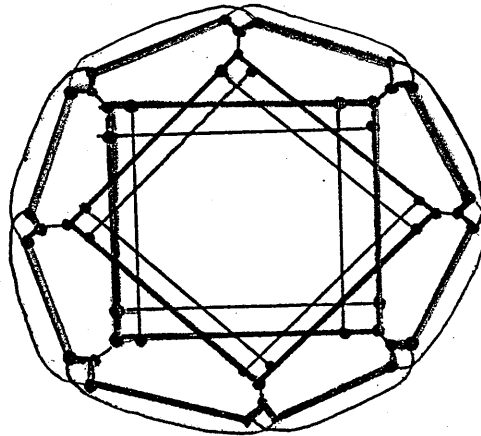


Figure 10.37

Case 6.  $n/(n,k) = 3$ . These graphs are all Class 2 because each inner component is the 3-critical subgraph of the Petersen graph.

Case 7.  $n/(n,k)$  is odd and at least 5, and  $n$  is even. Since  $n$  is even and  $n/(n,k)$  is odd,  $(n,k)$  must be even. We obtain an odd length path in each inner component  $I_i$  ( $0 \leq i \leq (n,k) - 1$ ). Each path includes all the vertices of the component  $I_i$  for  $i = 0, \dots, (n,k) - 1$ . We have an even number of components. We connect pairs of these  $I_i$  with two odd length paths in the outer rim, to obtain  $\frac{1}{2}(n,k)$  even cycles. We form an even cycle with the remaining vertices of the outer rim.

For each inner component  $I_i$  we take the path  $L_{i+3k'i}^1$   
 $0 \leq i \leq (n,k)$ . In the outer rim we take the  $\frac{1}{2}(n,k)$  paths  $P_{2i}$  and the  
 $\frac{1}{2}(n,k)$  paths  $P_{3k+2i}$ , for  $0 \leq i \leq \frac{1}{2}(n,k)-1$ . These paths combine to  
 give  $\frac{1}{2}(n,k)$  even cycles

$$P_{2i} L_{2i+1+3k,2i+1}^{-1} P_{3k+2i}^{-1} L_{2i+3k,2i}^1 \quad (0 \leq i \leq \frac{1}{2}(n,k)-1).$$

There are an even number  $3n - 4(n,k)$  vertices in the outer rim not in any  
 of the above cycles. The following is a cycle on these  $3n - 4(n,k)$   
 vertices (writing  $m = (n,k)$ ):

$$a_0 b_1 a_2 b_3 \dots b_{m-1} a_m c_m b_{m+1} \dots a_{3k} b_{3k+1} \dots b_{3k+m-1} a_{3k+m} c_{3k+m} b_{3k+m} \dots a_0.$$

All vertices are now in even cycles and hence  $D(n,k)$  is Class 1.

Example:  $D(10,2)$ .

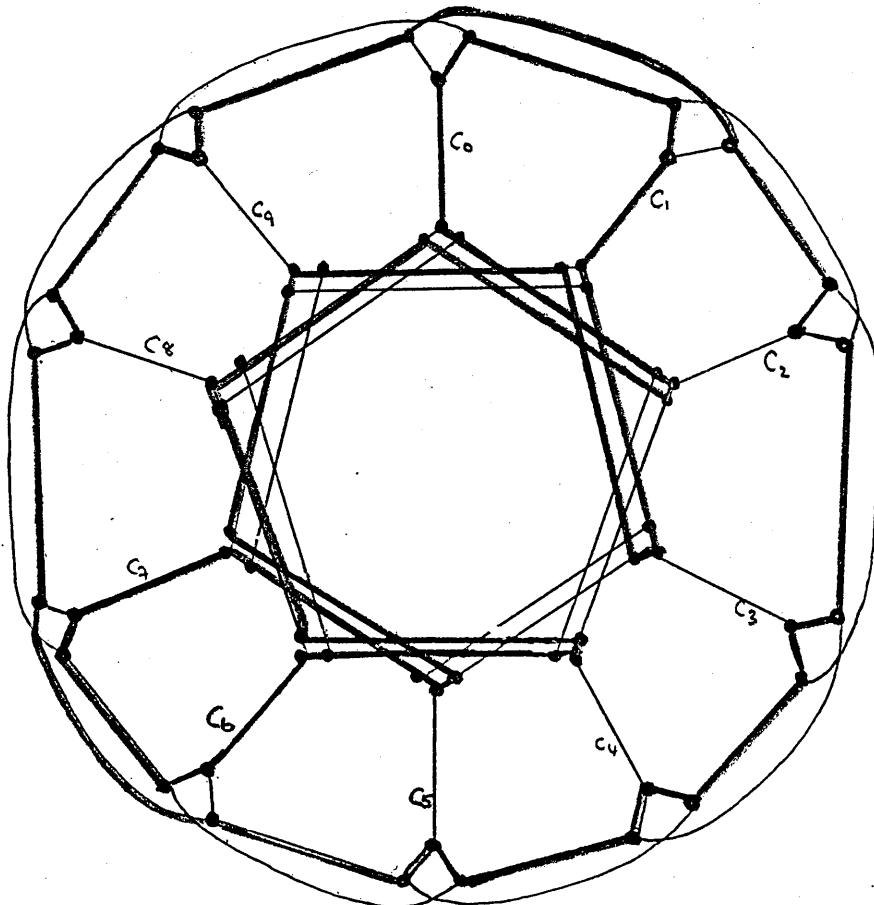


Figure 10.38

Case 8.  $n/(n,k)$  is odd and at least 5,  $n$  is odd and  $n \neq 2k + (n,k)$ .

Since  $n$  and  $n/(n,k)$  are both odd,  $(n,k)$  must also be odd.

In each inner component  $I_i$  ( $0 \leq i \leq (n,k)-1$ ), we obtain a path  $L'_{i+k, i+3k}$  of odd length and the remaining vertices form into even cycles of length 6. We join up all the paths of odd length with paths of odd length in the outer rim to form an even length cycle. The remaining outer vertices again form even cycles.

For  $I_i$  we have the path  $L'_{k+i, 3k+i}$  and the cycles  $O'_{i+5k}, O'_{i+7k}, \dots, O'_{i+n-2k}$ .

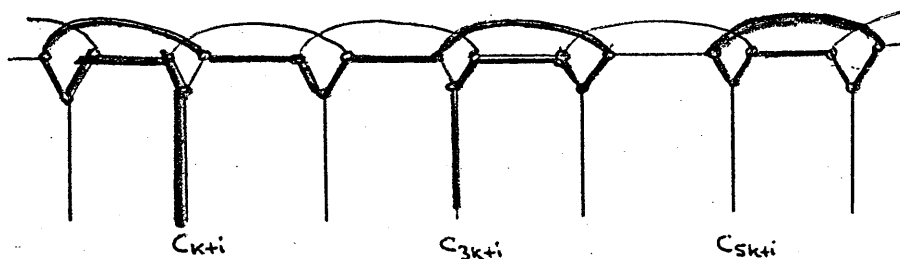


Figure 10.39

Then relative to the outer rim we have paths attached to the following spokes (writing  $m = (n,k)$ ):

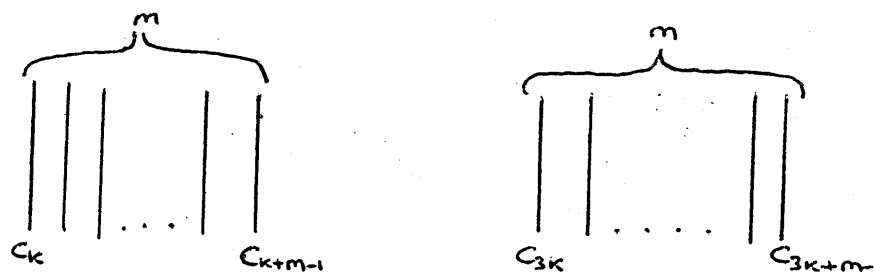


Figure 10.40

There are  $(n-m-2k)$  spokes between  $c_{k+n-1}$  and  $c_{3k}$ . Now  $n-m-2k$  is even; since  $k < \frac{1}{2}n$ ,  $n-m-2k \geq 0$ ; and since  $n \neq 2k + (n,k)$  we have that  $n-m-2k$  is even and at least 2. Hence we can link up the spokes in the following manner:

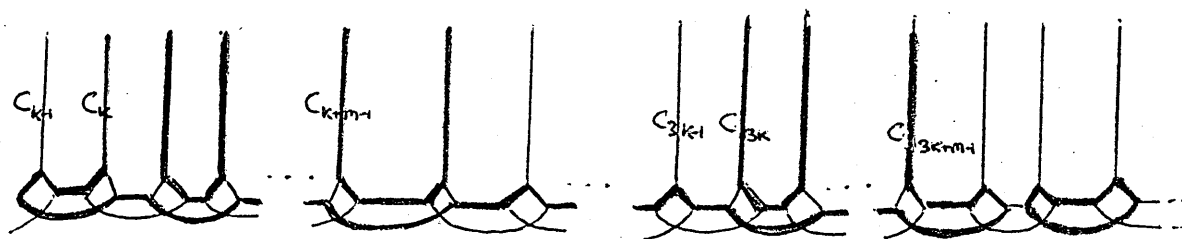


Figure 10.41

This path in the outer can be described by  $N_{k,k+m,3k-1,3k+m-1}$  with paths  $P_{k+2i+1}$ ,  $P_{3k+2i}$ , spokes  $c_{k+2i+1}$  to  $c_{k+2i+2}$  and  $c_{3k+2i}$  to  $c_{3k+2i+1}$  ( $0 \leq i \leq \frac{m-2}{2} - 1$ ) respectively.

The remaining cycles in the outer are  $O_{3k+m+2i+1}$  ( $0 \leq i \leq \frac{n-m-2k-2}{2}$ ).

Example:  $n = 15$ ,  $k = 3$ ,  $(n,k) = 3$ ,  $n/(n,k) = 5$ .

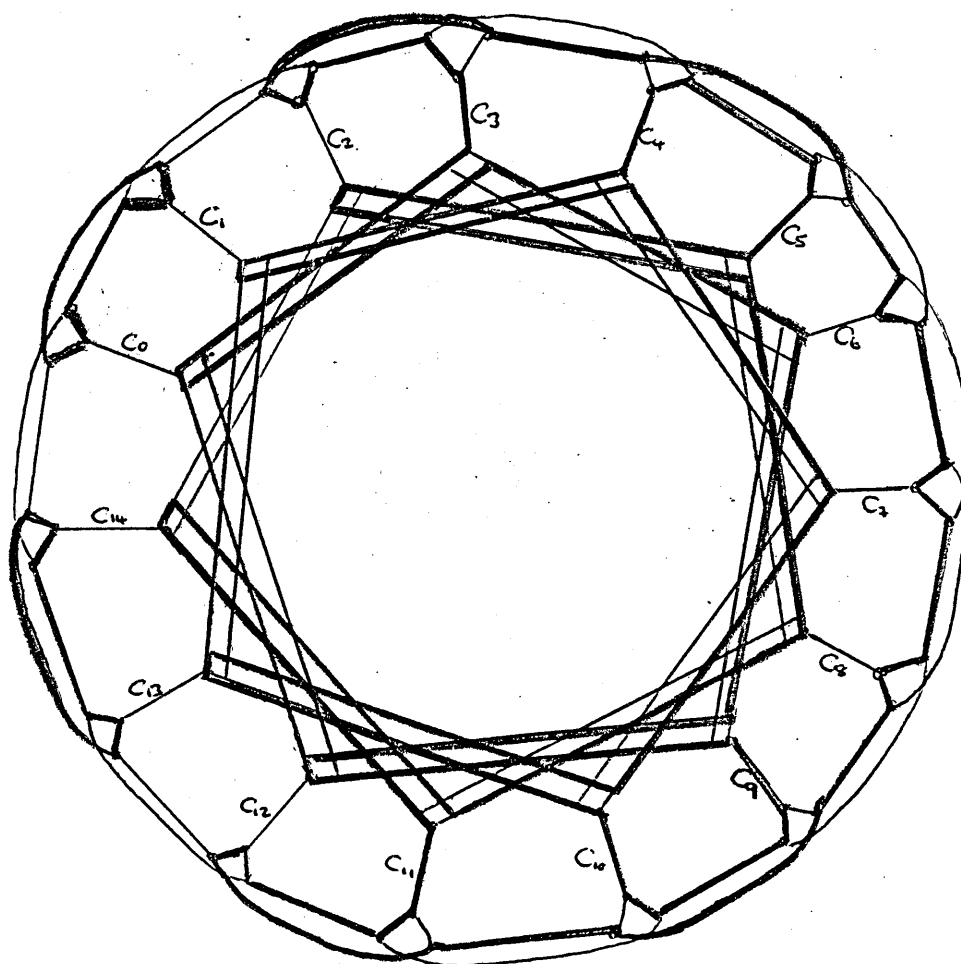


Figure 10.42

The long path in the outer rim is  $N_{3,6,9,11}$ .

Case 9.  $n/(n,k)$  is odd and at least 5,  $n$  is odd and  $n = 2k + (n,k)$ .

Let  $n = (2r+1)m$ , where  $m = (n,k)$ . We take paths in the inner components which join up with short paths in the outer rim as shown:

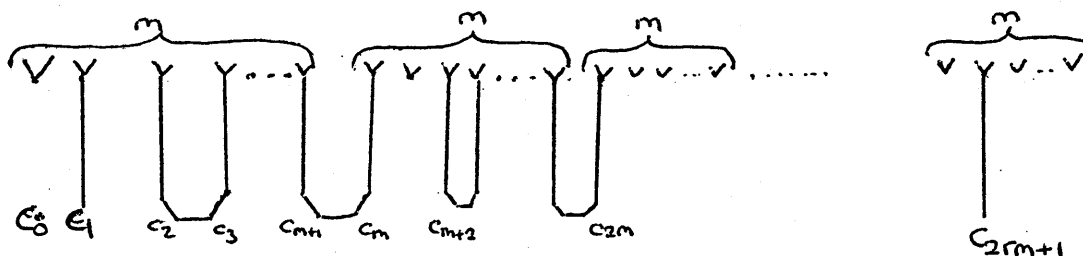


Figure 10.43

In  $I_0$  we take  $L'_{m,2m}$ ;

in  $I_1$  we take  $L'_{2rm+1,1}$ ,

and in

$I_i$  we take  $L'_{i,m+i} \quad (2 \leq i \leq m-1).$

We need paths  $P_{2i}, P_{m+2i} \quad (1 \leq i \leq \frac{m-1}{2}),$

and circuits  $0_{2rm+3}, 0_{2rm+5}, \dots, 0_{(2r+1)m-2}.$

The final path  $N_{1,m+1,m+2,2m+1,2rm+1}$  will combine with all the other odd paths to form one even circuit:

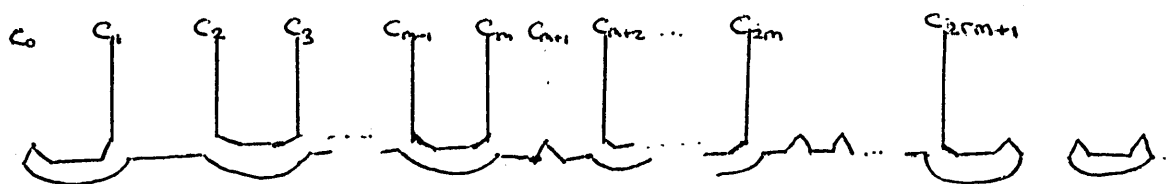


Figure 10.44

Example:  $n = 15$ ,  $k = 6$ ,  $(n,k) = m = 3$ .

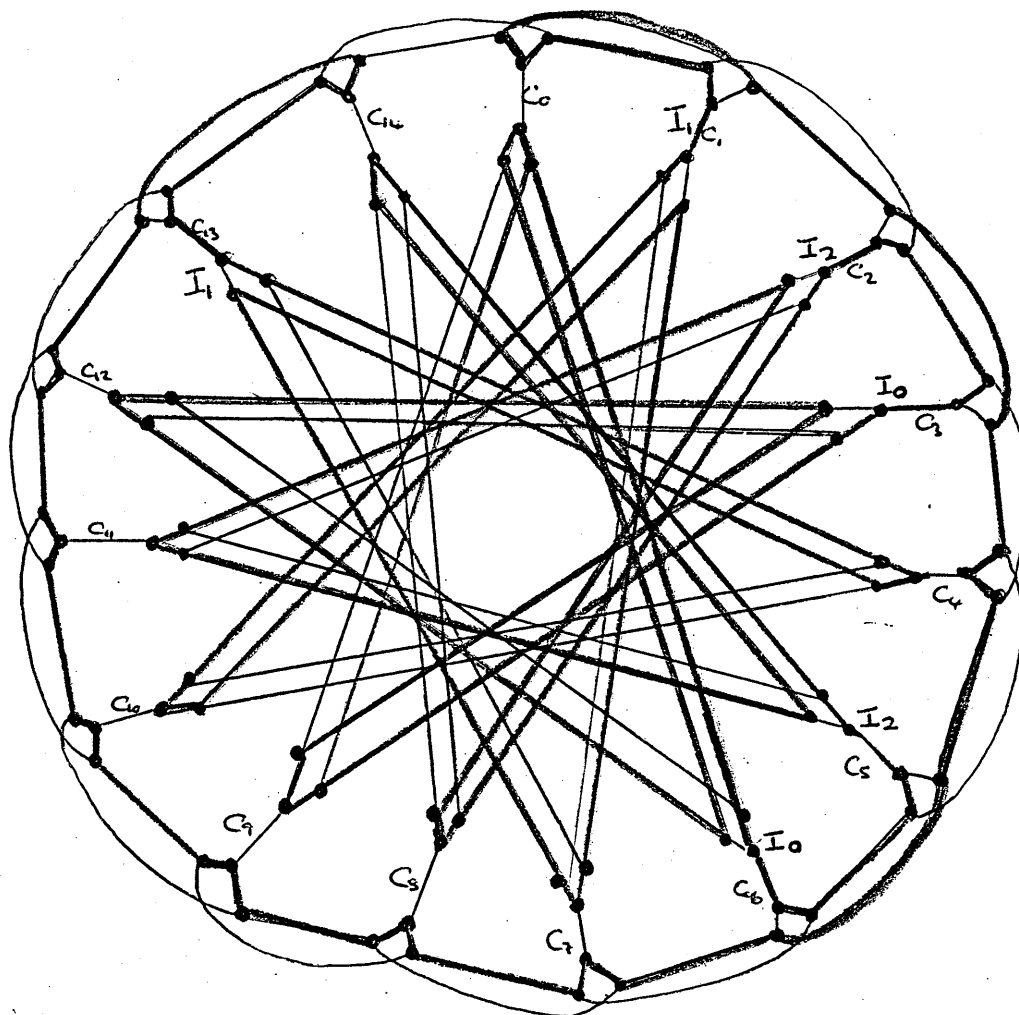


Figure 10.45

The long path in the outer rim is  $N_{1,4,7,12,13}$ .



## Chapter 11. Some interesting critical graphs.

In this chapter we collect together some critical graphs which are of interest for some reason or another.

### 11.1. A counterexample to Vizing's conjecture

We now give an example of a multigraph that does not satisfy Vizing's conjecture [V3] for the number of edges of a critical graph. [Vizing's conjecture was actually only stated for simple graphs.]

Vizing's Conjecture. If  $G$  is a  $k$ -critical graph with  $v$  vertices and  $e$  edges then  $e \geq \frac{1}{2}(kv - v + 3)$ . (Recall that Vizing's conjecture is also discussed in the introduction to Chapter 4 and after Conjecture 5.3). The following graph  $W$  is 7-critical with 9 vertices and 28 edges. From Vizing's conjecture, the number of edges should be at least 29.

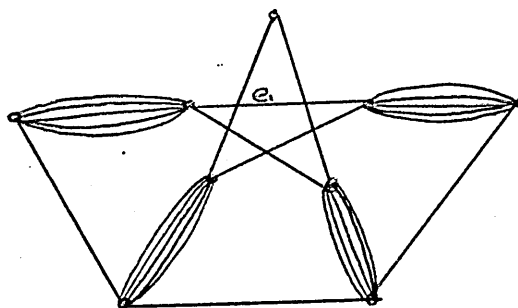


Figure 11.1

To establish this assertion, we first show that  $W$  is Class 2. Suppose  $W$  is Class 1. Since there are nine vertices there are  $\leq 4$  edges of each colour. Since there are 28 edges and 7 colours, each colour is on exactly 4 edges. So there are 7 near 1-factors whose union is  $W$ . Remove a near 1-factor containing  $e_1$  which includes an edge on each vertex of degree 7 (by symmetry, we need only consider one such near 1-factor). We now look for a near 1-factor including  $e_2$ . This is impossible and



### 11.2 The Critical Graph Conjecture

As remarked in Chapter 6 (see the discussion concerning Conjecture 6.3), Beineke and Wilson [ B2 ] and, independently, Jakobsen [ J4 ] made the following remark.

The Critical Graph Conjecture. Every critical graph has an odd number of vertices.

The evidence to support the conjecture was that there were no even order critical graphs on at most 10 vertices and no 3-critical graphs of even order on at most 16 vertices. The conjecture is now known to be false. The earliest counter-example was given in 1978 by M. Goldberg [ G3 ] who constructed an infinite sequence of 3-critical graphs of even order, the smallest of his examples having order 22.

We have found two 4-critical graphs of smaller orders. The graphs have orders 18 and 16. The graph of order 16 has a multiple edge. [I should point out that the conjecture was only stated for simple graphs]. Both graphs are drawn in Figure 11.4.

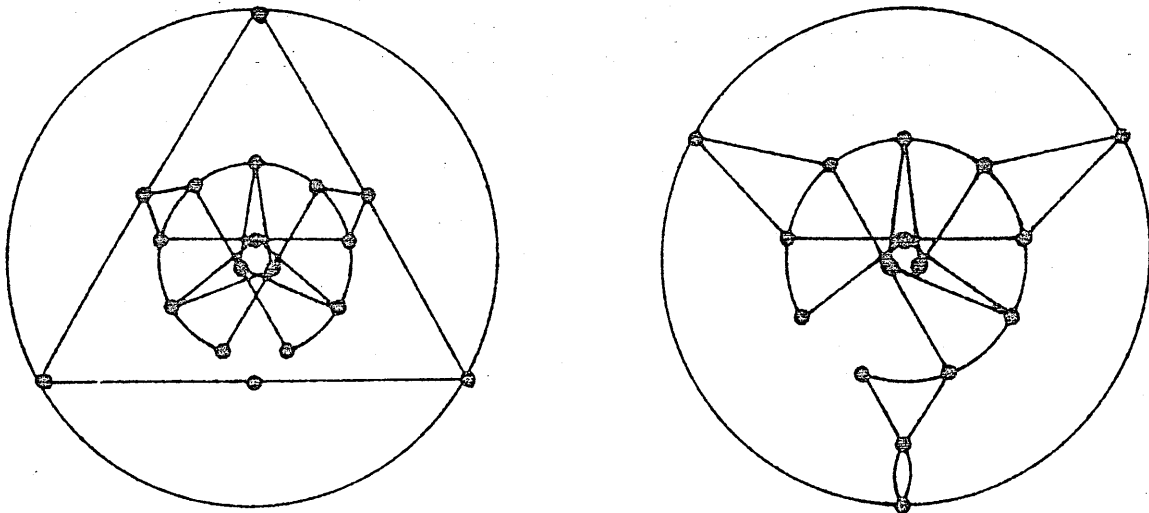


Figure 11.4

### 11.3. The graph that Yap used

Another graph which we found is drawn in Figure 11.5.

This graph is 4-critical of odd order and has 3 vertices of degree 2.

This graph is very useful since Yap [ Y1 ] had invented a construction for  $r$ -critical graphs of even order (for  $r \geq 4$ ) provided that there exists an  $r$ -critical graph  $G$  of odd order with a vertex of degree 2 and another vertex of degree at most  $\frac{1}{2}(r + 2)$ . Our graph is the only known example.

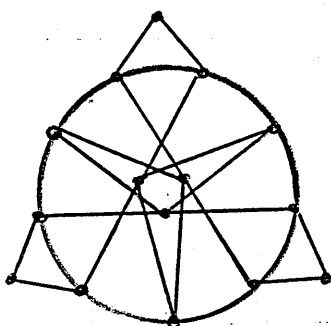


Figure 11.5

- A1. L. D. Andersen and A. J. W. Hilton, Generalized latin rectangles II: embedding, *Discrete Math.* 31 (1980), 235-260; MR 81#5020b.
- A2. K. Appel and W. Haken, Every planar map is four colourable, *Bull. Amer. Math. Soc.*, 82 (1976), 711-712; MR 54 #12561.
- B1. L. W. Beineke and S. Fiorini, On small graphs critical with respect to edge-colourings, *Discrete Math.* 16 (1976), 109-121; MR 55#2631.
- B2. L. W. Beineke and R. J. Wilson, On the edge-chromatic number of a graph, *Discrete Math.* 5 (1973), 15-20; MR 47#4836.
- B3 C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam (1973); MR 52#5453.
- B4. D. Blanusa, Problem ceteriju boja (The problem of four colours), *Hrvatsko Prirodoslovno Društvo Glasnik Mat.-Fiz. Astr. Ser. II* 1 (1946), 31-42; MR 10 #136e.
- B5. B. Bollobas and S. E. Eldridge, Maximal matchings in graphs with given maximal and maximal degrees, *Math. Proc. Camb. Phil. Soc.* 79 (1976), 221-234; MR 52#10491.
- C1. P. Cameron, A. G. Chetwynd and J. Watkins, The decomposition of Snarks, preprint.
- C2. F. Castagna and G. Prins, Every generalised Petersen graph has a Tait colouring, *Pacific J. Math.* 40 (1972), 53-58; MR 46#3358.

- C3. U. A. Celmins and E. R. Swart, The Construction of Snarks,  
*Department of Combinatorics and Optimization, University of Waterloo*  
*research report CORR 79-18.*
- C4. U. A. Celmins, A study of three conjectures on an infinite family  
of snarks, preprint.
- C5. A. G. Chetwynd and A. J. W. Hilton, Partial edge-colourings of  
complete graphs or of graphs which are nearly complete,  
*Proc. of conference at Cambridge (1983), to appear.*
- C6. A. G. Chetwynd and A. J. W. Hilton, The chromatic index of graphs  
of even order with many edges, *J. Graph Theory*, to appear.
- C7. A. G. Chetwynd and A. J. W. Hilton, The chromatic class of graphs  
with a given order and a given number of vertices of maximum  
degree, preprint.
- C8. A. G. Chetwynd and R. J. Wilson, Snarks and Supersnarks, *The*  
*Theory and Applications of Graphs*, Wiley, New York, (1981).
- C9. A. G. Chetwynd and H. P. Yap, Chromatic index critical graphs of  
order 9, *Discrete Math.* 47 (1983), 23-33.
- C10. V. Chvatal, On Hamilton's ideals, *J. Comb. Theory (B)* 12 (1972),  
163-168; MR 45#3228.
- D1. B. Descartes, Network colourings, *Math. Gazette* 32 (1948),  
67-69; MR 10#136d.

- D2. G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* (3) 2 (1952), 69-81; MR 13#856.
- F1. R. J. Faudree and J. Sheehan, Regular graphs and edge-chromatic number, *Discrete Math.* 48 (1984), 197-204.
- F2. M. A. Fiol, *Contribucion a la teoria de grafos regulares*, Barcelona University (1979).
- F3. M. A. Fiol, *3-grafos criticos*, Doctoral Dissertation, Barcelona University, Spain (1980).
- F4. S. Fiorini, *The chromatic index of simple graphs*, Doctoral dissertation, The Open University, England, (1974).
- F5. S. Fiorini, Some remarks on a paper by Vizing on critical graphs, *Math. Proc. Camb. Phil. Soc.* 77 (1975), 475-483; MR 51#10146.
- F6. S. Fiorini and R. J. Wilson, *Edge-Colorings of Graphs*, Research Notes in Mathematics 16, Pitman, London (1977); MR 58 #27599.
- F7. J. Fouquet, Note sur la non existence d'un "snark" ayant exactement 16 sommets, *C.N.R.S.*, Paris (1979).
- G1. M. Gardner, Mathematical games, *Scientific American*, 234 (1976), 126-130.

- G2. M. K. Goldberg, Construction of multigraphs with a constraint on the chromatic class, *Diskrēt. Analiz* 30. *Metody Diskrēt. Anal. v Rensenii Kombinatornyh Zadac* (1977), 3-12, 76; MR 58 #27643.
- G3. M. K. Goldberg, Trivalent graphs with chromatic index 4 (Russian), *Soobšč. Akad. Nauk Gruzii, SSR* 93 (1979), 29-31 MR 80 m#5039.
- G4. M. K. Goldberg, Construction of class 2 graphs with maximum vertex degree 3, *J. Comb. Theory (B)* 31 (1981), 282-291.
- H1. R. Häggkvist, Abstracts on Graph Theory from the Silver Jubilee conference on combinatorics (1982).
- H2. R. Häggkvist, Abstracts of Southampton Conference on combinatorics (1983).
- H3. S. L. Hakimi, On realizability of a set of integers as degrees of the vertices of a linear graph. *I. J. Soc. Indust. Appl. Math.* 10 (1962), 496-506; MR 26 #5558.
- H4. P. Hall, On representatives of subsets, *Jour. London Math. Soc.* 10 (1935), 26-30.
- H5. A. J. W. Hilton, Definitions of criticality with respect to edge-colouring, *J. Graph Theory* 1 (1977), 61-68; MR 56 #15479.



- H6. A. J. Hoffman and H. W. Kuhn, Systems of distinct representatives and linear programming, *Amer. Math. Monthly* 63 (1956), 455-60; MR 18 #370.
  
- I1. R. Isaacs, Infinite families of non-trivial trivalent graphs which are not Tait colourable, *Amer. Math. Monthly* 82 (1975), 221-239; MR 52 #2940.
  
- I2. R. Isaacs, Loupekhine's snarks: A bifamily of non-Tait-colourable graphs, preprint.
  
- J1. B. Jackson, Hamilton cycles in regular 2-connected graphs, *J. Comb. Theory (B)*. 29 (1980), 24-46.
  
- J2. F. Jaeger, Sur l'indice chromatique du graph représentatif des arêtes d'un graphe régulier, *Discrete Math.* 9 (1974), 161-172; MR 50 #6908.
  
- J3. I. T. Jakobsen, On critical graphs with chromatic index 4, *Discrete Math.* 9 (1974), 265-276; MR 50 #161.
  
- J4. I. T. Jakobsen, Some remarks on the chromatic index of a graph, *Arch. Math. (Basel)* 24 (1983), 440-448; MR 48 #10874.
  
- K1. D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, *Math. Ann.* 77 (1916), 453-465.
  
- K2. A. Kotzig, On the theory of finite regular graphs of degree three and four, *Čas. Pěst. Mat.* 82 (1957), 76-92. MR 19 #876.

- M1. C. McDiarmid, The solution to a time-tabling problem, *J. Inst. Maths. Applies.* 9 (1972), 23-34; MR 45 #9668.
- M2. G. H. J. Meredith, Regular  $n$ -valent,  $n$ -connected, non-Hamiltonian, non- $n$ -edge-colourable graphs, *J. Comb. Theory (B)* 14 (1973), 55-60; MR 47 #65.
- O1. W. E. Opencomb, On the intricacy of combinatorial problems, *Discrete Math.* (to appear).
- P1. M. Plantholt, The chromatic index of graphs with a spanning star, *J. Graph Theory* 5 (1981), 5-13; MR 83f #5023.
- P2. M. Plantholt, On the chromatic index of graphs with large maximum degree, *Discrete Math.* 47 (1983), 91-96.
- P3. M. Preissmann, Snarks of order 18, *Discrete Math.* 42 (1982), 125-126.
- R1. A. Rosa and W. D. Wallis, Premature sets of 1-factors or how not to schedule round robin tournaments, *Discrete Appl. Math.* 4 (1982), 291-297.
- S1. G. Szekeres, Polyhedral decompositions of cubic graphs, *Bull. Austral. Math. Soc.* 8 (1973), 367-387; MR 48 #3785.

- T1. P. G. Tait, Remarks on the colouring of maps, *Proc. Royal Soc. Edin.* 10 (1880), S01-S03, 729.
- T2. W. T. Tutte, The factorization of linear graphs, *J. London Math. Soc.* 22 (1947), 107-111; MR 9#297.
- V1. V. G. Vizing, On an estimate of the chromatic class of a  $p$ -graph (Russian), *Diskret. Analiz.* 3 (1964), 25-30; MR 31 #856.
- V2. V. G. Vizing, Critical graphs with a given chromatic class (Russian), *Diskret. Analiz* 5 (1965), 9-17; MR 34 #101.
- V3. V. G. Vizing, Some unsolved problems in graph theory, *Uspekhi Mat. Nauk* 23 (1968), 117-134; *Russian Math. Surveys* 23 (1968), 125-142.
- W1. M. E. Watkins, A theorem on Tait colourings with an application to the generalized Petersen graphs, *J. Comb. Theory* (B) 6 (1969), 152-164; MR 38 #4360.
- W2. D. de Werra, Equitable colorations of graphs, *Rev. France. Rech. Oper.* 5 (1971), 3-8; MR 58 #27610.
- Y1. H. P. Yap, On the critical graph conjecture, *J. Graph Theory* 4 (1980), 309-314; MR 81i #5071.

- Y2. H. P. Yap, On graphs critical with respect to edge-colourings,  
*Discrete Math.* 37 (1981), 289-296.
- Y3. H. P. Yap, A construction of chromatic index critical graphs,  
*J. Graph Theory* 5 (1981), 159-163; MR 82 #05049.